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Hydrodynamic limits for the free Kawasaki dynamics of continuous particle systems

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Abstract

An infinite particle system of independent jumping particles is considered. Their constructions is recalled, further properties are derived, the relation with hierarchical equations, Poissonian analysis, and second quantization are discussed. The hydrodynamic limit for a general initial distribution satisfying a mixing condition is derived. The large time asymptotic is computed under an extra assumption.

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Contents

1 Introduction

Particle systems in the continuum are describing infinitely many particles located in the Euclidean space $\mathbb{R}^d$. One may equip these systems with different types of dynamics, deterministic as well as stochastic. First of all, we should mention the Hamiltonian dynamics and related problems concerning the derivation of kinetic equations for classical gases, see, e.g. [Sin89]. Another dynamics strongly motivated by physical applications, is the gradient diffusion of infinitely many particles [Fri87], [Spo86]. Note that in spite of serious efforts in both cases, the answers obtained till now are far from being complete.

In order to identify further reasonable types of random evolutions in the continuum, we may look at the well established types of dynamics on lattice gas systems. There are two important classes of Markov dynamics, namely, Glauber and Kawasaki. Both are constructed in such a way that a given Gibbs measure (equilibrium state) on the lattice becomes an invariant measure for the dynamics. The Glauber dynamics is a birth-and-death evolution of the lattice gas, contrary to the jump type evolution in the case of the Kawasaki dynamics. The latter stochastic dynamics are especially interesting for the study of hydrodynamic limits of interacting particle systems due to their a priori conservation law, the number of particles, and the resulting existence of a continuous family of invariant measures [DMP91]. Continuous versions of Glauber dynamics were introduced in [Glö81], [BCC02], [KL05], [KLR07]. Continuous analogs of the Kawasaki dynamics are random evolutions of particle systems in which individual particles jump in the space with rates leading to a Gibbs state in the continuum as an invariant measure [Glö81], [KLR07].

In this paper, we concentrate on stochastic dynamics where each particle performs a jump process independently of the others. This type of dynamics one might call independent jump process or free Kawasaki dynamics in the continuum. One of the aims of this paper is to demonstrate that already for the free Kawasaki dynamics the situation is technically non trivial and different from the gradient diffusions. Another aim is to lay a solid ground for future investigation of interacting systems.

In order to better understand the chosen framework, one has to recall

the underlying motivation. Infinite particle systems are introduced as an approximation for finite but very large systems. The underlying finite systems are systems of finitely many particles in a bounded subset (as, for example, a ball or a cube) of $\mathbb{R}^d$. Each particle independently performs a Markov jump process. This underlying jump process we call in the following the one particle dynamics. Motivated by physics, one assumes that the particles are indistinguishable. Therefore, we describe the collection of positions of n particles not by a tuple $(x_1, \ldots, x_n)$ with $x_i \in \mathbb{R}^d$, but by the set $\{x_1, \ldots, x_n\}$ of points (excluding a priori coinciding positions) or by the integer valued measure $\delta_{x_1} + \ldots + \delta_{x_n}$. These notations include the indistinguishability automatically. We call this symmetrized collection of positions a configuration of particles. The latter interpretation, via measures, is called the empirical field. A key idea in the study of large particle systems is to substitute them by infinite volume systems in order to avoid boundary and finite size effects. The price to pay for this substitution is technical subtleties and difficulties, which we will describe in the sequel.

Let us consider an initial configuration of positions for the infinite particle system, denoted by γ. In any finite observation window, γ should look like a configuration of a large, but finite,particle system looked at in the same window. In the notation for configurations as sets this just means that for any open bounded subset $\Lambda \subset \mathbb{R}^d$ there can be only finitely many points in $\gamma \cap \Lambda$. In the interpretation of configurations as non-negative integer valued measures, this means that γ is a Radon measure. The construction of such a type of processes for general underlying one particle Markov process is given in [KLR09] and we shortly recall this construction in Subsection 3.1. In our case, each particle is equipped with an independent identically distributed exponential clock. If the clock of a particle rings the particle performs a random jump independent of the other particles. As one considers infinitely many particles, the construction is non trivial. First of all, the process cannot be started in any initial configuration. Actually, one may easily construct initial configurations for which the process explode, in the sense that infinitely many particles appear in a finite volume. Secondly, the infinite volume process is not any longer a jump process in the classical sense, because in each time interval a.s. infinitely many jump events occur. An essential step in the construction is to show that in any finite time interval only finitely many jump events are visible in a bounded observation window, i.e., the number of particles in the window stays finite and only finitely many particles pass the window. In Subsection 3.1 we add to the results of [KLR09] the description

of the path-space measure corresponding to the process. One easily sees that all Poisson measures (Poisson random fields) with constant intensities are invariant measures with respect to the free Kawasaki dynamics. If the one particle jump rate is symmetric than these measures are also reversible.

Till now we spoke only about processes starting in a single configuration of particles. Choosing an initial configuration at random w.r.t. a probability measure on the configuration space, one can derive further classes of processes. If, for example, one distributes the initial configurations w.r.t. an invariant measure, then one obtains a so-called equilibrium process. For the equilibrium processes w.r.t. reversible measures powerful methods of Dirichlet forms may be applied. In particular, existence can be shown even for general interacting systems, cf. [KLR07]. In the case of the free Kawasaki process, one can apply in addition second quantization techniques, cf. Section 4.1 which give a full description of the L^2-theory, also for non-symmetric jump rates.

One speaks of a non-equilibrium process if the initial distribution is not an invariant measure. For the interacting non-equilibrium Kawasaki dynamics even the existence of such a Markov process is unknown. As was mentioned above, the free non-equilibrium Kawasaki dynamics exists for a large class of initial configurations. The aim of this paper is to go beyond mere existence. We want to study the ergodic properties of the process, i.e., we consider the large time asymptotic of the process in the sense of limiting distributions. As a next step, we consider the so-called hydrodynamic limit for the free Kawasaki dynamics.

We shall expect that the difficulties to treat non-equilibrium processes depend on the class of initial distributions. Poisson measures for non-constant intensities form a class which seems to be the easiest to be handled in the case of the free Kawasaki dynamics. These distributions are no longer invariant measures, they are, in general, not equivalent to any invariant measure and there exists no semigroup theory for the free Kawasaki dynamics in L^p-spaces with respect to these measures, cf. Remark 5.1.

Note that the one-dimensional distributions of the dynamics can be completely described in terms of the one-dimensional distributions of the underlying one particle jump process. Here a delicate moment is that the initial distribution of this one particle process is the intensity function for the starting Poisson measure. A natural class of intensities to consider should thus include intensities corresponding to the invariant measures, i.e., all constant intensities. In the infinite volume, this prevents us to work just with inte-

grable intensities which from a probabilistic point of view would be natural for the one particle process. A right framework seems to be the cone of all non-negative measurable bounded intensities, but it contains intensities for which there seems to be no asymptotic, cf. Remark 5.8. The study of the large time asymptotic of the infinite particle processes reduces to the study of the large time asymptotic of the one-particle process for these initial intensities.

The non-ergodicity of the infinite particle processes is reflected in the non ergodicity of the one particle processes in this class of initial intensities, see Subsection 5.1. Under the additional assumption that the Fourier transform of the initial intensity is a signed measure we can show, applying Fourier analysis, that all processes with Poissonian initial distribution with such intensities have again a Poissonian distribution as the large time asymptotic, but with a constant intensity. This constant we can identify as the arithmetic mean of the initial intensity. For this result a careful consideration of the notion of convergence is required. This is one of the aforementioned technical pitfalls caused by infinite volume. The above condition on the Fourier transform of the initial intensity cannot be directly described in terms of the intensity itself. It seems natural to claim that the large time asymptotic exists for all bounded intensities which have arithmetic mean. However, for example, bounded slowly oscillating intensities may not have arithmetic mean, cf. Remark 5.8. Under the condition of weak local asymptotic normality of the one-particle process, in [DSS82] Theorem 4.1 the authors show that any free infinite particle dynamics is asymptotically similar to Brownian motion. We can show that that one-particle process is weak locally normal if the jump rate has finite second moments. Their proof is based on a central limit theorem type result and hence the existence of second moments is essential. The proof given in this paper is straight forward, simple and hold for general jump rates, but for a more restricted class of intensities.

In Subsection 5.2 the so-called hydrodynamic limit for the free Kawasaki dynamics is considered. This means that the asymptotic of the corresponding scaled empirical field $n_t^{(\varepsilon)}(\varphi, \mathbf{X}) := n(\varepsilon^d \varphi(\varepsilon\cdot), \mathbf{X}_{\varepsilon^{-\kappa}t})$, $\kappa = 1, 2$, is the object under study. In order to obtain a nontrivial asymptotic, the initial intensity z has to be scaled as $z(\varepsilon\cdot)$. The limiting process for $\varepsilon \to 0^+$ is a deterministic evolving density profile given as the solution of a partial differential equation. For symmetric jump rates the obtained limiting equation, for $\kappa = 2$, is a second order elliptic equation and the coefficients are given by the second moments of the jump distribution. For asymmetric jump rates, the limiting

differential equation, for $\kappa = 1$, is of first order, cf. Proposition 5.15 and Remark 5.16. One may obtain a combined equation if one considers weak asymmetries, cf. Proposition 5.17 and Remark 5.18. Actually, the limit gives a way to construct a solution for the partial differential equation. If the Fourier transform of the initial intensity is a measure, then these results can be obtained directly via Fourier transform, considering carefully the sense of convergence. If the jump rate has all moments finite, we are able to prove very strong convergence of the semi-group and hence can treat more general initial intensities. However, this is technically harder, because further careful approximations are necessary and a stronger sense of convergence has to be considered. The presentation of this result is postponed to Proposition 5.19 for the clarity of the presentation. In [DSS82] the authors also consideres the Case 1 of Proposition 5.15 and 5.19 for jump rates with finite second moment. In Proposition 5.15 only finite first moments of the jump rate are required, but not all intensities are considered.

The above described results about large time asymptotic and hydrodynamic limit are extended to general initial distributions in Section 6. We require some kind of mixing property for the initial distributions, the so-called decay of correlation, which we formulate in terms of the cumulants (or Ursell functions). For the hydrodynamic limit we have to require in addition that the first correlation function of the initial measure converges reasonably. The first correlation function is the density of the first moment measure of the empirical field. In the hydrodynamic limit the solutions of the associated partial differential equations are constructed as the limits of the first correlation functions, cf. Proposition 6.1. The large time asymptotic is clearly again a Poisson measure for a constant intensity. However, the constant is now the arithmetic mean of the first correlation function of the initial distribution.

We prove in Section 6.3 that the conditions required above are fulfilled, in particular, by Gibbs measures in the high temperature low activity regime.

2 Kawasaki dynamics

2.1 The generator

The configuration space $\Gamma := \Gamma_{\mathbb{R}^d}$ over $\mathbb{R}^d$, $d \in \mathbb{N}$, is defined as the set of all locally finite subsets of $\mathbb{R}^d$,

$$\Gamma := \left\{ \gamma \subset \mathbb{R}^d : |\gamma_\Lambda| < \infty \text{ for every compact } \Lambda \subset \mathbb{R}^d \right\},$$

where $|\cdot|$ denotes the cardinality of a set and $\gamma_\Lambda := \gamma \cap \Lambda$. As usual we identify each $\gamma \in \Gamma$ with the non-negative Radon measure $\sum_{x\in\gamma} \delta_x \in \mathcal{M}(\mathbb{R}^d)$, where δ_x is the Dirac measure with mass at x, $\sum_{x\in\emptyset} \delta_x$ is, by definition, the zero measure, and $\mathcal{M}(\mathbb{R}^d)$ denotes the space of all non-negative Radon measures on the Borel σ-algebra $\mathcal{B}(\mathbb{R}^d)$. This identification allows to endow Γ with the topology induced by the vague topology on $\mathcal{M}(\mathbb{R}^d)$, i.e., the weakest topology on Γ with respect to which all mappings

$$\Gamma \ni \gamma \longmapsto \langle f, \gamma\rangle := \int_{\mathbb{R}^d} \gamma(dx)\, f(x) = \sum_{x\in\gamma} f(x), \quad f \in C_c(\mathbb{R}^d),$$

are continuous. Here $C_c(\mathbb{R}^d)$ denotes the set of all continuous functions on $\mathbb{R}^d$ with compact support. By $\mathcal{B}(\Gamma)$ we will denote the corresponding Borel σ-algebra on Γ.

Given a non negative function $a \in L^1(\mathbb{R}^d, dx)$, the generator $L := L_a$ of the free Kawasaki dynamics for an infinite particle system is given by the informal expression

$$(LF)(\gamma) := \sum_{x\in\gamma} \int_{\mathbb{R}^d} dy\, a(x-y)\, (F(\gamma \setminus x \cup y) - F(\gamma)) \,. \tag{1}$$

We proceed to give a rigorous meaning to the right-hand side of (1). Let $\mathcal{O}_c(\mathbb{R}^d)$ denote the set of all open sets in $\mathbb{R}^d$ with compact closure. A $\mathcal{B}(\Gamma)$-measurable function F is called cylinder and exponentially bounded whenever there is a $\Lambda \in \mathcal{O}_c(\mathbb{R}^d)$ such that $F(\gamma) = F(\gamma_\Lambda)$ for all $\gamma \in \Gamma$, and $|F(\gamma)| \leq Ce^{c|\gamma_\Lambda|}$, $\gamma \in \Gamma$, for some $C, c > 0$. For such a function F one has

$$\begin{aligned} &\sum_{x\in\gamma} \int dy\, a(x-y)\, |F(\gamma \setminus x \cup y) - F(\gamma)| \\ \leq\ & \left[|\gamma_\Lambda| \int_{\mathbb{R}^d} dy\, a(y) + \int_\Lambda dy \sum_{x\in\gamma} a(x-y) \right] 2Ce^{c(|\gamma_\Lambda|+1)}, \end{aligned} \tag{2}$$

which is finite provided the configuration γ is an element in $\Gamma_a \subset \Gamma$,

$$\Gamma_a := \left\{ \gamma \in \Gamma : y \mapsto \sum_{x\in\gamma} a(x-y) \text{ is } L^1_{\mathrm{loc}}(\mathbb{R}^d, dy) \right\}.$$

In this case, the sum and the integral in (1) are finite, and thus the operator L is well-defined on the space $\mathcal{F}L^0_{eb}(\Gamma_a)$ of all cylinder functions exponentially bounded on Γ restricted to Γ_a.

Concerning the set Γ_a, we note that $\mu(\Gamma_a) = 1$ for any probability measure μ on Γ with first correlation function $k_\mu^{(1)}$ bounded. This follows from the fact that for each closed ball $B(n) \subset \mathbb{R}^d$ centered at 0 and radius $n \in \mathbb{N}$ we have

$$\begin{aligned}\int_\Gamma \mu(d\gamma) \int_{B(n)} dy \sum_{x\in\gamma} a(x-y) &= \int_{\mathbb{R}^d} dx\, k_\mu^{(1)}(x) \int_{B(n)} dy\, a(x-y)\\ &\leq C\|a\|_{L^1(\mathbb{R}^d,dx)} \mathrm{vol}(B(n))\end{aligned}$$

for any constant $C \geq |k_\mu^{(1)}|$. Here $\mathrm{vol}(B(n))$ denotes the volume of $B(n)$ with respect to the Lebesgue measure on $\mathbb{R}^d$. Clearly, the latter implies that $\mu(\Gamma \setminus \Gamma_a) = 0$.

In view of these considerations, throughout this work we shall restrict our setting to Γ_a.

Among the elements in $\mathcal{F}L^0_{eb}(\Gamma_a)$ we distinguish the functions $e_B(f)$, called Bogoliubov exponentials,

$$e_B(f,\gamma) := \prod_{x\in\gamma}(1+f(x)), \quad \gamma \in \Gamma,$$

for f being any bounded $\mathcal{B}(\mathbb{R}^d)$-measurable function with compact support ($f \in B_c(\mathbb{R}^d)$). A first reason to distinguish these functions is due to the especially simple form for the action of L on them, namely, for all $f \in B_c(\mathbb{R}^d)$ and all $\gamma \in \Gamma_a$,

$$\begin{aligned}(Le_B(f))(\gamma) &= \sum_{x\in\gamma} \int_{\mathbb{R}^d} dy\, a(x-y)\,(f(y)-f(x))\, e_B(f,\gamma\setminus x)\\ &= \sum_{x\in\gamma} (Af)(x) e_B(f,\gamma\setminus x),\end{aligned} \tag{3}$$

where

$$Af(x) := \int_{\mathbb{R}^d} dy\, a(x-y)\,(f(y)-f(x))\,. \tag{4}$$

In the sequel we call the linear operator A the one particle operator. Due to its special role throughout this work, its properties will be studied in more detail in the next subsection.

Given a locally integrable function $z \geq 0$, we remind that the Poisson measure π_z with intensity z is the unique probability measure on Γ for which

the Laplace transform is given by

$$\int_{\Gamma} \pi_z(d\gamma)\, e^{\langle\varphi,\gamma\rangle} = \exp\left(\int_{\mathbb{R}^d} dx\, \left(e^{\varphi(x)} - 1\right) z(x)\right), \tag{5}$$

for all φ in the Schwartz space $\mathcal{D}(\mathbb{R}^d) := C_c^\infty(\mathbb{R}^d)$ of all infinitely differentiable functions with compact support. We recall that a measure μ on Γ_a is called infinitesimally reversible with respect to the operator L defined in (1), whenever L is symmetric in $L^2(\Gamma_a, \mu)$.

Lemma 2.1 *Assume that a is an even function. Then for any real number $z > 0$ the Poisson measure π_z with constant intensity z is an infinitesimally reversible measure with respect to L.*

Proof. For all $F, G \in \mathcal{F}L^0_{eb}(\Gamma_a)$ we have

$$\begin{aligned}\int_{\Gamma_a} \pi_z(d\gamma) LF(\gamma)G(\gamma) \quad &= \quad \int_{\Gamma_a} \pi_z(d\gamma) \sum_{x\in\gamma} \int_{\mathbb{R}^d} dy\, a(y-x) F(\gamma \setminus x \cup y) G(\gamma) \\ &\quad - \int_{\Gamma_a} \pi_z(d\gamma) \sum_{x\in\gamma} \int_{\mathbb{R}^d} dy\, a(x-y) F(\gamma) G(\gamma).\end{aligned}$$

The result follows by applying twice the Mecke identity (cf. [Mec67, Theorem 3.1]) to the first integral. ■

Remark 2.2 It is clear that the linear space spanned by the class of functions $e_B(f)$, $f \in B_c(\mathbb{R}^d)$, is in $\mathcal{F}L^0_{eb}(\Gamma_a)$. The space $\mathcal{F}L^0_{eb}(\Gamma_a)$ also contains the class of coherent states $e_{\pi_z}(f)$ corresponding to $f \in B_c(\mathbb{R}^d)$,

$$e_{\pi_z}(f) := \exp\left(-\int_{\mathbb{R}^d} dx\, z(x) f(x)\right) e_B(f). \tag{6}$$

2.2 The one particle operator

In the sequel the one particle operator A introduced in (4) will play an essential role. Because of this, in this subsection we shall collect its main properties used below. We observe that in stochastic analysis the operator A is known as a relatively simple example of a bounded generator of a Markov jump process on $\mathbb{R}^d$ (see e.g. [EK86, Section 4.2]).

In terms of the usual convolution $*$ of functions, we also note that

$$Af = a * f - a^{(0)} f, \quad a^{(0)} := \int_{\mathbb{R}^d} a(x)\, dx. \tag{7}$$

Therefore, the properties of convolution of functions (namely, Young's inequality) lead straightforwardly to the L^p results stated in the next proposition. There, we also consider the real Banach spaces $B(\mathbb{R}^d)$ and $C_\infty(\mathbb{R}^d)$, respectively, of all bounded measurable functions and of all continuous functions vanishing at infinity, both with the supremum norm $\|f\|_u := \sup_{x\in\mathbb{R}^d} |f(x)|$. We recall that a strongly continuous contraction semigroup $(T_t)_{t\geq 0}$ is called sub-Markovian whenever for $0 \leq f \leq 1$ it follows $0 \leq T_t f \leq 1$ for every $t \geq 0$. If, in addition, $T_t f_n \nearrow 1$ for some sequence $f_n \nearrow 1$, then $(T_t)_{t\geq 0}$ is called Markovian. Here, for the spaces $B(\mathbb{R}^d)$ and $C_\infty(\mathbb{R}^d)$ the convergence is pointwise, and for an L^p-space the convergence is almost everywhere. A strongly continuous contraction semigroup $(T_t)_{t\geq 0}$ is called positivity preserving, if $f \geq 0$ implies $T_t f \geq 0$ for every $t \geq 0$.

Proposition 2.3 *The linear operator A is a bounded operator on $L^p(\mathbb{R}^d, zdx)$, for $z > 0$ constant and $1 \leq p \leq \infty$ (on $B(\mathbb{R}^d)$ and on $C_\infty(\mathbb{R}^d)$). As a consequence, A is the generator of a uniformly continuous semigroup $(e^{tA})_{t\geq 0}$ on $L^p(\mathbb{R}^d, zdx)$, $1 \leq p \leq \infty$ (on $B(\mathbb{R}^d)$ and on $C_\infty(\mathbb{R}^d)$),*

$$e^{tA} := \sum_{n=0}^{\infty} \frac{(tA)^n}{n!}, \tag{8}$$

the sum converging in norm for every $t \geq 0$. Moreover, $(e^{tA})_{t\geq 0}$ is a positivity preserving (contraction) semigroup on each $L^p(\mathbb{R}^d, zdx)$ space, $1 \leq p \leq \infty$ (on $B(\mathbb{R}^d)$ and on $C_\infty(\mathbb{R}^d)$), Markovian on $L^p(\mathbb{R}^d, zdx)$ for $1 \leq p < \infty$

For the proof see e.g. [EK86, Section 4.2], [Jac01].

Due to (7), the operator A, as well as the semigroup $(e^{tA})_{t\geq 0}$, both either on a $L^p(\mathbb{R}^d, zdx)$ space, $1 \leq p < \infty$, or on $C_\infty(\mathbb{R}^d)$, may be expressed in terms of the Fourier transform,

$$\hat{f}(k) := \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} dx\, e^{-i\langle x,k\rangle} f(x)$$

see e.g. [Jac01].

Proposition 2.4 *For every function φ in the Schwartz space $\mathcal{S}(\mathbb{R}^d)$ of tempered test functions one has*

$$\widehat{A\varphi}(k) = (2\pi)^{d/2}\left(\hat{a}(k) - \hat{a}(0)\right)\hat{\varphi}(k), \quad k \in \mathbb{R}^d,$$

and

$$\left(e^{tA}\varphi\right)(x) = \frac{1}{(2\pi)^{d/2}}\int_{\mathbb{R}^d} dk\, e^{i\langle k,x\rangle} e^{t(2\pi)^{d/2}(\hat{a}(k)-\hat{a}(0))}\hat{\varphi}(k), \quad x \in \mathbb{R}^d. \tag{9}$$

Remark 2.5 Since a is non-negative, for all $k \in \mathbb{R}^d$ one has

$$\mathrm{Re}(\hat{a}(k) - \hat{a}(0)) = \frac{1}{(2\pi)^{d/2}}\int_{\mathbb{R}^d} dx\,(\cos(\langle k,x\rangle) - 1)a(x) \leq 0. \tag{10}$$

The equality in (10) holds only for $k = 0$. This follows from the fact that the set $\{x : \langle k,x\rangle = 2n\pi, n \in \mathbb{Z}\}$ has zero Lebesgue measure if and only if $k \neq 0$.

Remark 2.6 According to Proposition 2.4, the semigroup $(e^{tA})_{t\geq 0}$ is defined by a kernel μ_t, $t \geq 0$, whose Fourier transform is given by

$$\hat{\mu}_t(k) = \frac{1}{(2\pi)^{d/2}} e^{t(2\pi)^{d/2}(\hat{a}(k)-\hat{a}(0))}.$$

Because a is non-negative, $\hat{a}$ and thus $\hat{\mu}_t$, $t \geq 0$, are positive definite functions. Through Bochner's theorem, the latter means that each μ_t is a non-negative finite measure on $\mathbb{R}^d$. We note that the Markovian property of $(e^{tA})_{t\geq 0}$ means that each μ_t is actually a probability measure. For more details see e.g. [Jac01].

Remark 2.7 It is easy to check that on the space $L^2(\mathbb{R}^d, zdx)$ the adjoint operator A^* of A is defined by

$$(A^*f)(x) = \int_{\mathbb{R}^d} dy\, a(y-x)(f(y)-f(x)) + \int_{\mathbb{R}} dy(a(y)-a(-y))\, f(x), \quad f \in L^2(\mathbb{R}^d, zdx).$$

As a consequence, if a is an even function, then A is a bounded self-adjoint operator on $L^2(\mathbb{R}^d, zdx)$. In this case, it follows from (8) that the semigroup $(e^{tA})_{t\geq 0}$ is a self-adjoint contraction on $L^2(\mathbb{R}^d, zdx)$.

Remark 2.8 For a non-constant activity parameter z, the operator A is still bounded but, in general, the semigroup $(e^{tA})_{t\geq 0}$ is not any longer a contraction. For instance, on $\mathbb{R}$ consider $a(x) = e^{-x^2}$ and $z(x) = 1 + e^{-x^2+4x}$. A simple calculation shows that for $\varphi(x) = \pi^{-1/2}e^{-x^2}$ one has $\int_{\mathbb{R}} dx\,\varphi(x)(A\varphi)(x)z(x) > 0$, proving that the semigroup $(e^{tA})_{t\geq 0}$ cannot be a contraction in $L^2(\mathbb{R}^d, zdx)$.

3 Independent infinite particle processes

3.1 A probabilistic approach

As we mentioned at the beginning of Subsection 2.2, the operator A is the generator of a Markov jump process on $\mathbb{R}^d$. Following e.g. [EK86], we can explicitly construct this process as follows. Consider the Markov chain $(Y_k)_{k\in\mathbb{N}_0}$ on $\mathbb{R}^d$, with transition density function $\mu(x,y) = \frac{a(x-y)}{a^{(0)}}$. Let $(Z_t)_{t\geq 0}$ be a Poisson process with parameter $a^{(0)}$ independent of the Markov chain. We then define the Markov process $(X_t)_{t\geq 0}$ by $X_t := Y_{Z_t}$, $t \geq 0$. This process has generator A and, by construction, it has *cadlag* paths in $\mathbb{R}^d$. We denote by $D([0,\infty),\mathbb{R}^d)$ the set of all *cadlag* paths from $[0,\infty)$ to $\mathbb{R}^d$ and by P^x the path-space measure corresponding to the process X starting at $x \in \mathbb{R}^d$. By E_x we denote the expectation w.r.t. this measure.

The process on Γ corresponding to L is the following random evolution: each particle evolves according to the above jump process, independently of the other particles, cf. Lemma 3.1 below. This independent infinite particle process was rigorously constructed in [KLR09] (see also [KLR03]). In the following we present the main results therein.

Assume that there exist an $\alpha > 2d$ and a $C > 0$ such that

$$0 \leq a(y) \leq \frac{C}{(1+|y|)^\alpha}, \qquad \text{for all } y \in \mathbb{R}^d. \tag{11}$$

The condition sufficient for the construction done in [KLR09] is then fulfilled. Hence there exists a Markov process $(D([0,\infty),\Theta),(\mathbf{X}_t)_{t\geq 0},(\mathbf{P}_\gamma)_{\gamma\in\Theta})$ on the set Θ of all $\gamma \in \Gamma$ such that, for some $m \in \mathbb{N}$ (depending on γ),

$$|\gamma_{B(n)}| \leq m \operatorname{vol}(B(n)), \quad \forall\, n \in \mathbb{N}.$$

Here $D([0,\infty),\Theta)$ denotes the set of all *cadlag* paths from $[0,\infty)$ to Θ, the process $\mathbf{X}_t : D([0,\infty),\Theta) \to \Theta$ is the canonical one, i.e., $\mathbf{X}_t(\omega) := \omega(t)$, $\omega \in D([0,\infty),\Theta)$, and each $\mathbf{P}_\gamma$ is the path-space measure of the process starting at a $\gamma \in \Theta$. By $\mathbf{E}_\gamma$ we denote the expectation w.r.t. $\mathbf{P}_\gamma$.

Note that the process cannot start at any arbitrary initial configuration $\gamma \in \Gamma$, as follows implicitly from the discussion before Theorem 2.2 in [KLR09]. One is obliged to restrict the set of possible initial configurations e.g. to Θ. However, by [KKK04], [Kun99], one has $\mu(\Theta) = 1$ for every probability measure μ on Γ whose correlation functions $k_\mu^{(n)}$, $n \in \mathbb{N}$, fulfill the

so-called Ruelle bound, i.e., there is a $C > 0$ such that $k_\mu^{(n)} \leq C^n$ for every $n \in \mathbb{N}$. This holds, for instance, for Gibbs measures w.r.t. superstable, lower and upper regular potentials, cf. [Rue70]. In particular, it holds for Poisson measures with intensity being any non-negative bounded measurable function. For a constant intensity this result was shown using ergodicity in [NZ76].

Choosing, for a fixed initial configuration $\gamma \in \Theta$, an enumeration $\{x_n\}_{n\in\mathbb{N}}$, the infinite particle process can be described more explicitly. For each n let us consider an independent copy of the one-particle jump process $((X_t^{(n)})_{t\geq 0}, P^{x_n})$ introduced at the beginning of the section. In [KLR09] it is shown that a.s. the sequence $(X_t^{(n)})_{n\in\mathbb{N}}$ has no accumulation points and no two points in it which coincide. Then $(\{X_t^{(n)}\}_{n\in\mathbb{N}})_{t\geq 0}$ is the corresponding process on the configuration space Θ and the path-space measure is the "symmetrization" of $\bigotimes_{n=1}^\infty P^{x_n}$. Moreover, the transition probability $(\mathbf{P}_t)_{t\geq 0}$ of the process $(\mathbf{X}_t)_{t\geq 0}$ is just the product of the one-particle transition probabilities $e^{tA}(x-y)dy$, i.e., $\prod_{n=1}^\infty e^{tA}(x_n - y_n)dy_n$. As a consequence, for all non-positive $\varphi \in \mathcal{D}(\mathbb{R}^d)$ we have

$$\mathbf{E}_\gamma\left[e^{\langle\varphi,\mathbf{X}_t\rangle}\right] = \int_\Theta \mathbf{P}_t(\gamma, d\xi)\, e^{\langle\varphi,\xi\rangle} = \prod_{x\in\gamma} E_x\left[e^{\varphi(X_t)}\right], \tag{12}$$

cf. [KLR09] (see also Lemma 3.2 below), or in terms of Bogoliubov exponentials for $\varphi \in \mathcal{D}(\mathbb{R}^d)$ with $-1 < \varphi \leq 0$,

$$\mathbf{E}_\gamma\left[e_B(\varphi, \mathbf{X}_t)\right] = \prod_{x\in\gamma} E_x\left[\varphi(X_t) + 1\right] = e_B(e^{tA}\varphi, \gamma). \tag{13}$$

More generally, it follows that

$$\mathbf{E}_\gamma\left[e^{\langle\varphi_1,\mathbf{X}_{t_1}\rangle}\cdots e^{\langle\varphi_n,\mathbf{X}_{t_n}\rangle}\right] = \prod_{x\in\gamma} E_x\left[e^{\varphi_1(X_{t_1})}\cdots e^{\varphi_n(X_{t_n})}\right],$$

for all $0 \leq t_1 < \ldots < t_n$ and all non-positive $\varphi_1, \ldots, \varphi_n \in \mathcal{D}(\mathbb{R}^d)$, $n \geq 2$. By a monotone approximation procedure using Riemannian sums, this relation allows us to calculate the Laplace transform of the path-space measure $\mathbf{P}_\gamma$, i.e.,

$$\mathbf{E}_\gamma\left[e^{-\int dt\langle\varphi(t,\cdot),\mathbf{X}_t\rangle}\right] = \prod_{x\in\gamma} E_x\left[e^{-\int dt\varphi(t,X_t)}\right], \tag{14}$$

for all non-negative continuous functions φ from $[0,\infty)\times\mathbb{R}^d$ to $\mathbb{R}$ with compact support.

The next result gives a relation between the process $\mathbf{X}$ and L.

Lemma 3.1 *For all $F \in \mathcal{F}L^0_{eb}(\Theta)$ and all $\gamma\in\Theta$ it holds*

$$\mathbf{E}_\gamma\left[F(\mathbf{X}_t)-F(\mathbf{X}_0)-\int_0^t ds LF(\mathbf{X}_s)\right]=0. \tag{15}$$

Proof. For all F of the form $e_B(\varphi)$ with $\varphi\in\mathcal{D}(\mathbb{R}^d)$ one has

$$\frac{d}{dt}\mathbf{E}_\gamma\left[F(\mathbf{X}_t)\right]=\frac{d}{dt}e_B(e^{tA}\varphi,\gamma)=\sum_{x\in\gamma}Ae^{tA}\varphi(x)e_B(e^{tA}\varphi,\gamma\setminus x).$$

Hence using the product structure of the path-space measure and (3) one finds

$$\sum_{x\in\gamma}Ae^{tA}\varphi(x)e_B(e^{tA}\varphi,\gamma\setminus x)=\mathbf{E}_\gamma\left[\sum_{x\in\mathbf{X}_t}A\varphi(x)e_B(\varphi,\mathbf{X}_t\setminus x)\right]=\mathbf{E}_\gamma\left[Le_B(\varphi)(\mathbf{X}_t)\right].$$

In order to make the previous calculations rigorous and to extend them to the whole space $\mathcal{F}L^0_{eb}(\Theta)$, we have to establish bounds for the considered expressions. According to (2), for all $F\in\mathcal{F}L^0_{eb}(\Theta)$, one may bound the integrand $LF(\mathbf{X}_s)$ in (15) by

$$\left[|\mathbf{X}_s\cap\Lambda|\int_{\mathbb{R}^d}dy\,a(y)+\int_\Lambda dy\sum_{x\in\mathbf{X}_s}a(x-y)\right]2Ce^{c(|\mathbf{X}_s\cap\Lambda|+1)}.$$

For some constants $C',C''>0$ one may bound the previous expression by $C''e^{\langle\chi,\mathbf{X_s}\rangle}$, where $\chi(y)=\frac{C'}{(1+|y|)^{\alpha/2}}$ with α as in (11). The first summand in (15) and the expression $\sum_{x\in\mathbf{X}_t}|A\varphi(x)|e_B(|\varphi|,\mathbf{X}_t\setminus x)$ can be bounded analogously. Due to Lemma 3.2 below the expectation of these bounds are finite for all $\gamma\in\Theta$. ■

Lemma 3.2 *Let $\chi(y)=\frac{C'}{(1+|y|)^{\alpha/2}}$, for some $C'>0$ and $\alpha>0$ as in (11). Then there exists a $c>0$ such that $A\chi\le c\chi$ and $e^{tA}\chi\le e^{ct}\chi$ for all $t\ge0$. Moreover, for all $\gamma\in\Theta$ and all measurable functions φ such that $\frac{|\varphi|}{\chi}$ is bounded, one has that the product $\prod_{x\in\gamma}(1+\varphi(x))$ is absolutely convergent and*

$$\mathbf{E}_\gamma\left[e^{\langle\varphi,\mathbf{X}_t\rangle}\right]<\infty.$$

Proof. The previous considerations yield

$$\mathbf{E}_\gamma\left[e^{\langle\varphi,\mathbf{X}_t\rangle}\right] \le \mathbf{E}_\gamma\left[e^{\langle\chi,\mathbf{X}_t\rangle}\right] = \prod_{x\in\gamma} E_x\left[e^{\chi(X_t)}\right] = \prod_{x\in\gamma} e^{tA}e^{\chi}(x).$$

Thus the proof amounts to show the convergence of the latter infinite product. For this purpose, we note that

$$\chi(x-y) = \frac{C'}{(1+|x-y|)^{\alpha/2}} \le \frac{C'(1+|y|)^{\alpha/2}}{(1+|x|)^{\alpha/2}} = (1+|y|)^{\alpha/2}\chi(x).$$

This gives for the one particle operator

$$A\chi(x) = \int_{\mathbb{R}^d} dy a(y)\chi(x-y) - a^{(0)}\chi(x) \le \Big(\int_{\mathbb{R}^d} dy(1+|y|)^{\alpha/2}a(y)\Big)\chi(x) - a^{(0)}\chi(x).$$

Due to the bound (11) for a, the integral $\int_{\mathbb{R}^d} dy(1+|y|)^{\alpha/2}a(y)$ is finite. By Grönwall's lemma this yields $E_x[\chi(X_t)] = e^{tA}\chi(x) \le \chi(x)e^{ct}$, where $c = \int_{\mathbb{R}^d} dy(1+|y|)^{\alpha/2}a(y) - a^{(0)}$. Therefore, also $|e^{tA}(e^{\chi}-1)|$ decays as $(1+|y|)^{-\alpha/2}$. According to the definition of Θ, one can prove that $\langle\chi,\gamma\rangle < \infty$ for all $\gamma\in\Theta$. Hence also the product $\prod_{x\in\gamma} e^{tA}e^{\chi}(x)$ converges absolutely and is finite. ■

In Sections 5 and 6 one considers the one-dimensional distributions of processes starting with initial distributions μ which are probability measures on Θ. The path-space measure $\mathbf{P}^\mu$ corresponding to such a process is given by $\int_\Theta \mathbf{P}_\gamma\mu(d\gamma)$. Its one-dimensional distribution is a probability measure $P^{\mathbf{X}}_{\mu,t}$ on Θ defined for all non-negative measurable functions F by

$$\int_\Theta P^{\mathbf{X}}_{\mu,t}(d\gamma)F(\gamma) := \int_\Theta \mu(d\gamma)\mathbf{E}_\gamma\left[F(\mathbf{X}_t)\right]. \tag{16}$$

In particular, for $\mu = \delta_\gamma$, $\gamma\in\Theta$, the one-dimensional distribution coincides with the transition kernel $\mathbf{P}_t(\gamma,\cdot)$ described above.

For functions F being Bogoliubov exponentials, definition (16) leads to the so-called Bogoliubov functionals [Bog46]. By definition, the Bogoliubov functional corresponding to a probability measure μ on Γ is defined by

$$B_\mu(\varphi) := \int_\Gamma e_B(\varphi,\gamma)\mu(d\gamma), \qquad \varphi\in\mathcal{D}(\mathbb{R}^d).$$

Such a functional is an analogue of the Fourier-Laplace transform on configuration spaces, cf. [Kun09]. Due to (13) there is an interesting relation between

the Bogoliubov functional corresponding to the initial distribution and the Bogoliubov functional corresponding to the one-dimensional distribution of the process at a time $t > 0$, namely,

$$\int_\Theta P^{\mathbf{X}}_{\mu,t}(d\gamma) e_B(\varphi,\gamma) = \int_\Theta \mu(d\gamma) e_B(e^{tA}\varphi,\gamma) = B_\mu(e^{tA}\varphi). \tag{17}$$

In particular, for $\mu = \pi_z$ for some bounded intensity function $z \geq 0$ one finds

$$\int_\Theta e_B(\varphi,\gamma) P^{\mathbf{X}}_{\pi_z,t}(d\gamma) = \int_\Theta e_B(e^{tA}\varphi,\gamma)\pi_z(d\gamma) = \exp\left(\int_{\mathbb{R}^d} e^{tA}\varphi(x) z(x)\,dx\right) \tag{18}$$

for all $\varphi \in \mathcal{D}(\mathbb{R}^d)$. In Section 5 we shall consider this special case in more detail.

3.2 An analytic approach

Within the framework of infinite dimensional analysis on configuration spaces [KK02] one may derive alternative representations and constructions of the dynamics. Instead of describing the infinite particle dynamics on Γ through the Kolmogorov equation $\frac{\partial}{\partial t} F_t = LF_t$, the so-called K-transform [Len75a, Len75b] allows an alternative description for the action of L on $\mathcal{F}L^0_{eb}(\Gamma_a)$.

Given the space $\Gamma_0 := \{\gamma \in \Gamma : |\gamma| < \infty\}$ of finite configurations endowed with the metrizable topology as described in [KK02], let $B_{exp,ls}(\Gamma_0)$ be the space of all exponentially bounded Borel measurable functions G (i.e., $|G(\eta)| \leq C_1 e^{C_2|\eta|}$ for some $C_1, C_2 > 0$) with local support (i.e., there is a $\Lambda \in \mathcal{O}_c(\mathbb{R}^d)$ such that $G\restriction_{\{\eta\in\Gamma_0 : |\eta\cap(\mathbb{R}^d\setminus\Lambda)|\neq 0\}} \equiv 0$). The K-transform of a $G \in B_{exp,ls}(\Gamma_0)$ is the mapping $KG : \Gamma \to \mathbb{R}$ defined for all $\gamma \in \Gamma$ by $(KG)(\gamma) := \sum_{\substack{\eta\subset\gamma\\|\eta|<\infty}} G(\eta)$. Note that $K\left(B_{exp,ls}(\Gamma_0)\right) \subset \mathcal{F}L^0_{eb}(\Gamma_a)$. Given a $G \in B_{exp,ls}(\Gamma_0)$, then in terms of the operator L we obtain

$$\begin{aligned}(L(KG))(\gamma) &= \sum_{x\in\gamma} \int_{\mathbb{R}^d\setminus(\gamma\setminus x)} dy\, a(x-y) \left[(KG)(\gamma\setminus x) + (KG(\cdot\cup y))(\gamma\setminus x)\right. \\ &\quad \left. -(KG)(\gamma\setminus x) - (KG(\cdot\cup x))(\gamma\setminus x)\right]\end{aligned}$$

which leads to the so-called symbol acting on quasi-observables $\hat{L}$,

$$(\hat{L}G)(\eta) := \sum_{x\in\eta} \int_{\mathbb{R}^d} dy\, a(x-y) \left(G(\eta\setminus x\cup y) - G(\eta)\right), \quad G \in B_{exp,ls}(\Gamma_0),$$

and the corresponding time evolution equation $\frac{\partial}{\partial t}G_t = \hat{L}G_t$. The transition kernel $\hat{P}_t$ corresponding to $\hat{L}$ is then given by

$$\int_{\Gamma_0} \hat{P}_t(\eta', d\eta)\, G(\eta) = \int_{\mathbb{R}^{dn}} \prod_{i=1}^{n} \int_{\mathbb{R}^d} dy_i\, e^{tA}(x_i - y_i) G(\{y_1, \ldots, y_n\}).$$

This allows us to extend the explicit formula for transition kernels of $\mathbf{X}$ to the class of all so-called observables of additive type

$$\int_{\Theta} \mathbf{P}_t(\gamma, d\xi)\, (KG)(\xi) = K\left(\int_{\Gamma_0} \hat{P}_t(\cdot, d\eta)\, G(\eta)\right)(\gamma).$$

By duality one may extend the dynamical description to correlation functions. For this purpose, on the space $\Gamma_0 = \cup_{n=0}^{\infty}\{\gamma \in \Gamma : |\gamma| = n\}$ let us consider the so-called Lebesgue-Poisson measure $\lambda_z := \sum_{n=0}^{\infty} \frac{1}{n!}(zm)^{(n)}$, where m denotes the Lebesgue measure and each $(zm)^{(n)}$, $n \in \mathbb{N}$, is the image measure on $\{\gamma \in \Gamma : |\gamma| = n\}$ of the product measure $z(x_1)dx_1 \cdots z(x_n)dx_n$ under the mapping $(\mathbb{R}^d)^n \ni (x_1, ..., x_n) \mapsto \{x_1, ..., x_n\}$. If for some probability measure μ on Γ there is a function k_μ on Γ_0 such that the equality $\int_\Gamma \mu(d\gamma)\, (KG)(\gamma) = \int_{\Gamma_0} \lambda(d\eta)\, G(\eta)k_\mu(\eta)$ holds for all $G \in B_{exp,ls}(\Gamma_0)$ we call k_μ the correlation function corresponding to μ. Here we abbreviate $\lambda := \lambda_1$. Denoting by $\hat{L}^*$ the dual operator of $\hat{L}$ in the sense

$$\int_{\Gamma_0} \lambda_z(d\eta)\, (\hat{L}G)(\eta)k(\eta) = \int_{\Gamma_0} \lambda_z(d\eta)\, G(\eta)(\hat{L}^*k)(\eta),$$

one obtains the following expression

$$(\hat{L}^*k)(\eta) = \sum_{x\in\eta} \int_{\mathbb{R}^d} dy\, a(y - x)k(\eta \setminus x \cup y) - |\eta|k(\eta)a^{(0)}.$$

The corresponding time evolution equation for correlation functions, $\frac{\partial}{\partial t}k_t = \hat{L}^*k_t$, is the analogue of the BBGKY-hierarchy for the case of the free Kawasaki dynamics. In our case this equation can be explicitly solved, namely,

$$k_t(\{x_1, \cdots, x_n\}) = \int_{\mathbb{R}^{dn}} dy_1 \cdots dy_n\, k_\mu(\{y_1, \cdots, y_n\}) \prod_{i=1}^{n} e^{tA}(y_i - x_i) \qquad (19)$$

for the initial condition k_μ. Let us note that if one assumes a Ruelle bound for the initial correlation function k_μ, then all the above considerations can

be made rigorous. Moreover, similar arguments used in [KKZ06] show that each k_t is actually a correlation measure corresponding to some probability measure μ_t on Γ.

According to the previous considerations, the time evolution of the particle system may also be described in terms of Bogoliubov functionals, cf. [KKO06], through the time evolution equation

$$\frac{\partial}{\partial t}B_{\mu,t}(\varphi) = \int_{\mathbb{R}^d} dx \int_{\mathbb{R}^d} dy\, a(x-y)(\varphi(y)-\varphi(x))\frac{\delta B_{\mu,t}(\varphi)}{\delta\varphi(x)}, \quad \varphi\in\mathcal{D}(\mathbb{R}^d).$$

Here $\frac{\delta B_{\mu,t}(\varphi)}{\delta\varphi(x)}$ denotes the first variational derivative of $B_{\mu,t}$ at φ. Actually, $B_{\mu,t}(\varphi)$ is the Bogoliubov functional corresponding to the one-dimensional distribution of the process starting in μ. Hence, the previous equation has an explicit solution given by (17), i.e., $B_{\mu,t}(\varphi) = B_\mu(e^{tA}\varphi)$.

4 Equilibrium dynamics

In this section we are interested in the representation of the generator L and its semigroup in terms of the creation, annihilation and second quantization operators as analytic expressions. These representations are possible because there is a well-known canonical unitary isomorphism between the (symmetric) Fock space and the Poisson space. We start by recalling this isomorphism. Our approach is based on [AKR98] and [KSSU98], but see also [KKO02] and references therein.

4.1 The $L^2(\pi_z)$ space and the Fock space representation

Let z be a non-negative constant. We consider the complex Hilbert space $L^2(\pi_z) := L^2(\Gamma,\mathcal{B}(\Gamma),\pi_z)$ of square integrable complex valued functions on Γ with respect to the Poisson measure π_z. The coherent states introduced in Remark 2.2 generate the system of so-called Charlier polynomials, namely,

$$e_{\pi_z}(\varphi,\gamma) = \sum_{n=0}^\infty \frac{1}{n!}\langle C_z^n(\gamma),\varphi^{\otimes n}\rangle, \qquad C_z^n(\gamma)\in(\mathcal{D}')^{\hat{\otimes}n},$$

where $(\mathcal{D}')^{\hat{\otimes}n}$ is the n-th symmetric tensor product of the Schwartz distributions space $\mathcal{D}'(\mathbb{R}^d)$. This system is orthogonal and any $F\in L^2(\pi_z)$ can be

expanded in terms of Charlier polynomials

$$F(\gamma)=\sum_{n=0}^{\infty}\langle C_z^n(\gamma),f^{(n)}\rangle,\qquad f^{(n)}\in L^2(zdx)^{\hat{\otimes}n}. \tag{20}$$

This yields a unitary isomorphism I_{π_z} between $L^2(\pi_z)$ and the so-called symmetric Fock space

$$\mathcal{F}(L^2(zdx)):=\bigoplus_{n=0}^{\infty}n!L^2(zdx)^{\hat{\otimes}n},\qquad L^2(zdx)^{\hat{\otimes}0}:=\mathbb{C}.$$

More precisely, for each $F\in L^2(\pi_z)$ of the form (20) one has $I_{\pi_z}(F)=(f^{(n)})_{n=0}^{\infty}$. Next we recall the definition of annihilation, creation and second quantization operators on the total subset of Fock space vectors of the form $f^{(n)}=f_1\hat{\otimes}\ldots\hat{\otimes}f_n$. The action of the annihilation operator $a^-(h)$ of a $h\in L^2(zdx)$ on $f^{(n)}$ is given by

$$a^-(h)f^{(n)}:=\sum_{j=1}^{n}(h,f_j)f_1\hat{\otimes}\ldots\hat{\otimes}f_{j-1}\hat{\otimes}f_{j+1}\hat{\otimes}\ldots\hat{\otimes}f_n\in L^2(zdx)^{\hat{\otimes}(n-1)}. \tag{21}$$

The adjoint operator of $a^-(h)$, called creation operator and denoted by $a^+(h)$, acts on elements $f^{(n)}\in L^2(zdx)^{\hat{\otimes}n}$ by $a^+(h)f^{(n)}=h\hat{\otimes}f^{(n)}$.

Given a contraction semigroup $(e^{tA})_{t\geq 0}$ on $L^2(zdx)$ one can construct a contraction semigroup $(\mathrm{Exp}(e^{tA}))_{t\geq 0}$ on $\mathcal{F}(L^2(zdx))$ defined by $e^{tA}\otimes\ldots\otimes e^{tA}$ on each space $L^2(zdx)^{\hat{\otimes}n}$. Its generator is the so-called second quantization operator $d\mathrm{Exp}A$ corresponding to A. Hence the image of the Fock coherent state $e(f):=(f^{\otimes n}/n!)_{n=0}^{\infty}$, $f\in L^2(zdx)$, under $\mathrm{Exp}(e^{tA})$ is given by

$$\mathrm{Exp}(e^{tA})(e(f))=e(e^{tA}f). \tag{22}$$

Through the unitary isomorphism I_{π_z} we obtain a contraction semigroup $(\mathrm{Exp}_{\pi_z}(e^{tA}))_{t\geq 0}$ on $L^2(\pi_z)$. In particular, since $I_{\pi_z}^{-1}e(f)=e_{\pi_z}(f)$, it follows from (6) that

$$\mathrm{Exp}_{\pi_z}\left(e^{tA}\right)e_B(f)=e_B(e^{tA}f). \tag{23}$$

In our case, for non-constant functions z the semi-group e^{tA} is in general not a contraction, see Remark 2.8 and indeed the previous construction ought to fail, cf. Remark 5.1.

Finally, we would like to present the "annihilation and creation operators" in the form more common in physical literature. In this heuristic way one can

also treat non-constant intensities. For each $x \in \mathbb{R}^d$ we define an operator $a^-(x)$ acting on $\vec{f} = (f^{(n)})_n$ by

$$(a^-(x)\vec{f})^{(n)}(y_1, \ldots, y_n) = \sqrt{n+1} f^{(n+1)}(x, y_1, \ldots, y_n).$$

The adjoint of the operator $a^-(x)$ is formally given by

$$(a^+(x)\vec{f})^{(n)}(y_1, \ldots, y_n) = \frac{1}{\sqrt{n}} \sum_{k=1}^{n} \delta(x - y_k) f^{(n-1)}(y_1, \ldots, \hat{y}_k, \ldots, y_n),$$

where $\hat{y}_k$ means that the k-th coordinate is excluded. Actually, the expression for $a^+(x)$ is well-defined as a quadratic form. It is easy to check the relations $a^{\pm}(f) = \int_{\mathbb{R}^d} a^{\pm}(x) f(x) dx$ in the sense of quadratic forms.

The operator L can be expressed in terms of creation and annihilation operators, namely,

$$L = d\mathrm{Exp}_{\pi_z}(A) + a^-_{\pi_z}(A^*_z 1), \tag{24}$$

where A^*_z is the adjoint operator of A with respect to the scalar product on $L^2(\mathbb{R}^d, z(x)dx)$, $d\mathrm{Exp}_{\pi_z}(A)$ and $a^-_{\pi_z}$ are, respectively, the image of $d\mathrm{Exp}(A)$ and a^- under I_{π_z}. For more details see e.g. [KKO02]. Note that the expression of the annihilation operator $a^-_{\pi_z}$ in $L^2(\pi_z)$ depends on the intensity of the underlying Poisson measure. Rewritten in a style more similar to the common usage in physics, (24) takes the form

$$\int dx\, z(x) \int dy \left(a(x-y) - a^{(0)} \delta(x-y) \right) \left(a^+_\pi(x) a^-_\pi(y) - a^-_\pi(y) \right),$$

where $a^{\pm}_\pi(x)$ are the image of $a^{\pm}(x)$ under I_{π_z}.

4.2 The symmetric case

If a is an even function, the situation is essentially simpler (see Lemma 2.1) and yields an alternative construction to the one presented in Section 3. As a matter of fact, for a an even function, and for each constant $z > 0$, the operator L defined in (1) gives rise to a Dirichlet form on $L^2(\Gamma, \pi_z)$,

$$\int_\Gamma \pi_z(d\gamma)(LF)(\gamma)F(\gamma) = -\frac{1}{2} \int_\Gamma \pi_z(d\gamma) \sum_{x \in \gamma} \int_{\mathbb{R}^d} dy\, a(x-y) \left| F(\gamma \setminus x \cup y) - F(\gamma) \right|^2.$$

This allows the use of Dirichlet forms techniques to derive a Markov process on Γ with *cadlag* paths and having π_z as an invariant measure [KLR07]. Actually, one can show that in this situation L is the second quantization operator corresponding to the non-positive self-adjoint operator A on $L^2(\mathbb{R}^d, zdx)$ (Remark 2.7). Hence, L is a negative essentially self-adjoint operator on $L^2(\Gamma, \pi_z)$, and it is the generator of a contraction semigroup on $L^2(\Gamma, \pi_z)$.

4.3 Second quantization operators

According to (13) and (6) the action of the transition kernel on coherent states corresponding π_z is given by

$$\mathbf{E}_\gamma \left[e_{\pi_z}(\varphi, \mathbf{X}_t) \right] = e_{\pi_z}(e^{tA}\varphi, \gamma) \exp\left(\int_{\mathbb{R}^d} dx \, \left(e^{tA}\varphi(x) - \varphi(x) \right) z \right). \tag{25}$$

This allows to express the action of the transition probability on coherent states in terms of annihilation and creation operators

$$\mathrm{Exp}_{\pi_z}(e^{tA}) e^{a^-_{\pi_z}(A_z^* 1)}. \tag{26}$$

Observe that the right-hand side of (25) gives the action of a semigroup which preserves coherent states. Lemma 4.1 shows that L is the generator of a strongly continuous Markov semigroup.

Lemma 4.1 *Let $\mathcal{D}_{\mathrm{coh}}$ be the vector space spanned by all functions $e_B(\varphi)$ with $\varphi \in L^1(\mathbb{R}^d, zdx) \cap L^2(\mathbb{R}^d, zdx)$. The operator L restricted to $\mathcal{D}_{\mathrm{coh}}$ is closable in $L^2(\Gamma, \pi_z)$ and its closure is an extension of the operator $(L, \mathcal{F}L^0_{\mathrm{eb}}(\Gamma_a))$ defined in Section 2. Moreover, it is the generator of a Markov semigroup and $e^{tL} = \mathrm{Exp}_{\pi_z}(e^{tA})$.*

The proof is an adaptation for a non-symmetric generator of the proof of Proposition 4.1 in [AKR98]. Technically, in our case the proof is simpler, because A is a bounded operator and we consider functions of the type $e_B(\varphi)$.

Proof. According to the bound (2), one can calculate the action of the adjoint of $(L, \mathcal{F}L^0_{\mathrm{eb}}(\Gamma_a))$ by the Mecke formula, i.e.,

$$(L^*F)(\gamma) := \sum_{x\in\gamma} \int_{\mathbb{R}^d} dy \, \left(a(y-x)F(\gamma \setminus x \cup y) - a(x-y)F(\gamma) \right), \quad \forall F \in \mathcal{F}L^0_{\mathrm{eb}}(\Gamma_a).$$

Hence L^* is densely defined and L is closable on any dense subset of $\mathcal{F}L^0_{\text{eb}}(\Gamma_a)$. Using again the bound (2) one sees that the closures of $(L, \mathcal{D}_{\text{coh}})$ and $(L, \mathcal{F}L^0_{\text{eb}}(\Gamma_a))$ coincide. As A is a bounded operator in $L^1(\mathbb{R}^d, zdx)$ and in $L^2(\mathbb{R}^d, zdx)$, the space $\mathcal{D}(\mathbb{R}^d)$ is a core of A in L^2 as well as in L^1. Define an $L^2(\Gamma, \pi_z)$ semigroup on $\mathcal{D}_{\text{coh}}$ (as an extension by continuity) of the following explicit formula

$$T_t e_B(\varphi) := e_B(e^{tA}\varphi).$$

Hence $\mathcal{D}_{\text{coh}}$ is a core for the generator of T_t. By computation one sees that $\varphi \mapsto e_B(\varphi)$ is a differentiable function from $L^1(\mathbb{R}^d, zdx) \cap L^2(\mathbb{R}^d, zdx)$ into $L^2(\Gamma, \pi_z)$ with derivative given by

$$\varphi \mapsto \sum_{x\in\gamma} \varphi(x) e_B(\varphi, \gamma \setminus x).$$

Summarizing, this yields that T_t is a strongly continuous contraction semigroup on $L^2(\Gamma, \pi_z)$ such that for all $\varphi \in L^1(\mathbb{R}^d, zdx) \cap L^2(\mathbb{R}^d, zdx)$ holds

$$\frac{d}{dt} T_t e_B(\varphi)(\gamma) = \sum_{x\in\gamma} A e^{tA}\varphi(x) e_B(e^{tA}\varphi, \gamma \setminus x).$$

For $\varphi \in \mathcal{D}(\mathbb{R}^d)$ the r.h.s. is just $Le_B(e^{tA}\varphi)$. Since A is bounded this can be extended to all $\varphi \in L^1(\mathbb{R}^d, zdx) \cap L^2(\mathbb{R}^d, zdx)$. These two facts combined yield $e^{tL} = T_t$ and $\mathcal{D}_{\text{coh}}$ is a core of L. Moreover, the operator L coincides with the second quantization operator $d\text{Exp}_{\pi_z}(A)$ on $\mathcal{D}_{\text{coh}}$. This space is also a core for $d\text{Exp}_{\pi_z}(A)$, cf. [BK88]. Hence the two operators coincide. ∎

5 Local equilibrium dynamics

The previous considerations were mainly for Poisson measures π_z which have as an activity parameter a constant function z. According to Lemma 2.1 and Section 2, such measures are reversible measures for the free Kawasaki process. As a first step towards other initial distributions, in this section we consider the so-called local equilibrium case, that are, Poisson measures with non constant activity parameter. In this case, if the activity z is a slowly varying function, then in a small volume the Poisson measure effectively has a constant activity. This property justifies the chosen name for this section.

Remark 5.1 Although for bounded non-constant functions z the process starting in π_z can be constructed, cf. Subsection 3.1, one cannot expect that the expression given in (17) can be extended to a semigroup either in the $L^1(\Gamma_a, \pi_z)$ or in the $L^2(\Gamma_a, \pi_z)$ sense. The existence of the semigroup in an L^p-sense can only be expected w.r.t. an invariant measure. This can be seen from the following equality

$$\frac{\|e^{tL}e_B(f)\|_{L^p(\pi_z)}}{\|e_B(f)\|_{L^p(\pi_z)}} = \exp\left(\frac{1}{p}\int_{\mathbb{R}^d}\left(|1+e^{tA}f(x)|^p - |1+f(x)|^p\right)z(x)dx\right),$$

and the fact that

$$\|e^{tL}\|_{\mathrm{Op},L^p(\pi_z)} \geq \sup_{f\in\mathcal{D}(\mathbb{R}^d), f\geq 0,} \exp\left(\frac{1}{p}\int_{\mathbb{R}^d}\left((1+e^{tA}f(x))^p - (1+f(x))^p\right)z(x)dx\right),$$

where the r.h.s. is infinite if $(e^{tA})_{t\geq 0}$ is not a contraction semigroup (see Remark 2.8).

Let $z \geq 0$ be a bounded measurable function. According to (18), for the one-dimensional distribution of the free Kawasaki process $(\mathbf{X}_t)_{t\geq 0}$ with initial distribution π_z one finds

$$\int_\Theta e_B(\varphi,\gamma)P^{\mathbf{X}}_{\pi_z,t}(d\gamma) = \exp\left(\int_{\mathbb{R}^d} dx\ e^{tA}\varphi(x)z(x)\right), \tag{27}$$

for every $\varphi \in \mathcal{D}(\mathbb{R}^d)$ and every $t \geq 0$.

For each $t \geq 0$ fixed, let us now consider the linear functional defined on $L^1(\mathbb{R}^d, dx)$ by

$$L^1(\mathbb{R}^d, dx) \ni f \mapsto \int_{\mathbb{R}^d} dx\, (e^{tA}f)(x)z(x).$$

Due to the contractivity property of the semigroup e^{tA} in $L^1(\mathbb{R}^d, dx)$, see Proposition 2.3, and to the boundedness of z, this functional is bounded on $L^1(\mathbb{R}^d, dx)$, and thus it is defined by a kernel $z_t \in L^\infty(\mathbb{R}^d, dx)$, that is,

$$\int_{\mathbb{R}^d} dx\, e^{tA}f(x)z(x) = \int_{\mathbb{R}^d} dx\, f(x)z_t(x), \tag{28}$$

for all $f \in L^1(\mathbb{R}^d, dx)$. Moreover, since e^{tA} is positivity preserving in $L^1(\mathbb{R}^d, dx)$, it follows from (28) that $z_t \geq 0$.

This shows by (27) that the one-dimensional distribution $P^{\mathbf{X}}_{\pi_z,t}$ is just the Poisson measure π_{z_t} (see also [Dob56], [Doo53]).

Furthermore, the path-space measure $\mathbf{P}^{\pi_z}$ corresponding to the process is also Poissonian. It is a Poisson measure on $\Gamma_{D([0,\infty),\mathbb{R}^d)}$ with intensity P^z, where P^z is the path-space measure on $D([0,\infty),\mathbb{R}^d)$ of the one-particle jump process corresponding to A with initial distribution $z(x)dx$. Using the fact that the path-space measure $\mathbf{P}_\gamma$ is supported on $D([0,\infty),\Theta)$ one easily sees that this implies that $\mathbf{P}^{\pi_z}$ is actually supported on a subset of $\Gamma_{D([0,\infty),\mathbb{R}^d)}$ which can be naturally identified with $D([0,\infty),\Theta)$.

Lemma 5.2 *The path-space measure $\mathbf{P}^{\pi_z}$ of the process $\mathbf{X}$ starting with initial distribution π_z is the Poisson measure π_{P^z} on $\Gamma_{D([0,\infty),\mathbb{R}^d)}$. In particular, for any continuous function φ on $[0,\infty)\times\mathbb{R}^d$ with compact support we have*

$$\int_{\Gamma_{D([0,\infty),\mathbb{R}^d)}} \mathbf{P}^{\pi_z}(d\omega)e^{-\int_0^\infty \langle\varphi(t,\cdot),\omega(t)\rangle dt} = \exp\left(\int_{D([0,\infty),\mathbb{R}^d)} \left[e^{-\int_0^\infty \varphi(t,\omega(t))dt}-1\right] P^z(d\omega)\right).$$

Proof. By the definition of the path-space measure corresponding to the initial measure π_z one has

$$\int_{\Gamma_{D([0,\infty),\mathbb{R}^d)}} \mathbf{P}^{\pi_z}(d\omega)e^{-\int_0^\infty \langle\varphi(t,\cdot),\omega(t)\rangle dt} = \int_\Theta \mathbf{E}_\gamma\left[e^{-\int_0^\infty\langle\varphi(t,\cdot),\mathbf{X}_t\rangle dt}\right]\pi_z(d\gamma).$$

Due to (14) the latter expectation is equal to

$$\int_\Theta e_B(E_\cdot\left[e^{-\int_0^\infty \varphi(t,X_t)dt}\right]-1,\gamma)\pi_z(d\gamma) = \exp\left(\int_{\mathbb{R}^d} E_x\left[e^{-\int_0^\infty\varphi(t,X_t)dt}-1\right]z(x)dx\right),$$

which yields the required result. ■

5.1 Large time asymptotic

In this subsection we want to study the behavior of the one-dimensional distribution of the free Kawasaki dynamics for large times. As mentioned above (28), the free Kawasaki dynamic leaves the Poissonian structure of the initial distribution unchanged and only the underlying intensity develops with time. Thus, the analysis of the large time asymptotic behavior reduces to the large time asymptotic of the intensity, cf. Lemma 5.9. In other words, we reduced the problem to a one particle problem where the intensity plays the role of the initial distribution. In the following remark we discuss which initial distributions are natural in this context.

Remark 5.3 As we work in the framework of a system scale much larger than the time scale, namely, we first perform the thermodynamic limit and only afterwards consider the time asymptotic, the situation is more subtle than it might seem at a first glance. The question is which class of initial distributions is the natural one. Usually, in the study of an one particle system one assumes that the initial data is integrable. In this case the dynamics is ergodic and the probability measure concentrated on the empty configuration is the invariant measure. Physically, this situation describes systems with zero density and perturbations of them. These perturbations are already singular in the sense that the corresponding Poisson measures are mutually singular. From a physical point of view, non-zero densities are interesting. The corresponding invariant measures are the Poisson measures with constant intensities $z > 0$. However, these intensities are not any longer integrable perturbations of the zero intensity. Furthermore, also their mutual differences are not integrable. Hence a natural setting for the initial intensities seems to be bounded non-negative measurable functions.

Definition 5.4 *One says that a function* $z \in L^1_{\mathrm{loc}}(\mathbb{R}^d, dx)$ *has arithmetic mean, denoted by* $\mathrm{mean}(z)$*, whenever the following limit exists*

$$\lim_{R\to+\infty} \frac{1}{\mathrm{vol}(B(R))} \int_{B(R)} dx\, z(x). \tag{29}$$

Note that not every bounded function has arithmetic mean, cf. Remark 5.8.

The following proposition states that the large time asymptotic of the one particle distribution is still Poissonian with intensity given by the arithmetic mean of the initial intensity.

Proposition 5.5 *Let* $z \geq 0$ *be a bounded measurable function whose Fourier transform is a signed measure. Then* z *has arithmetic mean and the one-dimensional distribution* $P^{\mathbf{X}}_{\pi_z,t}$ *converges weakly to* $\pi_{\mathrm{mean}(z)}$ *when* t *goes to infinity.*

This result is proved combining Lemma 5.9 and Corollary 5.12.

Note that, in particular, one sees that the process is not ergodic, because the large time asymptotic depends on the initial distribution. The situation of integrable intensities is technically comparable with the case in which the time scale is much larger than the space scale, e.g., instead of $\mathbb{R}^d$ one works on a torus.

Under the assumptions of Proposition 5.5, the activity z has arithmetic mean, cf. Corollary 5.12. Then, using Fourier transform, one can easily derive the large time asymptotic, cf. Lemma 5.10. The same technique also yields that the large time asymptotic only depends on the space asymptotic of the intensity z, cf. Corollary 5.14.

In Proposition 5.5, the assumption concerning the Fourier transform is actually not elegant, because we cannot reasonably restate it in terms of z in the position variables. Therefore, we give several examples below, cf. Example 5.7, and we derive certain properties of the arithmetic mean. Some types of reasonable asymptotic behavior do not fulfill the aforementioned Fourier transform assumption, e.g. Example 5.7(v).

Remark 5.6 If two functions $z_1, z_2 \in L^1_{\mathrm{loc}}(\mathbb{R}^d, dx)$ have arithmetic mean, then for every $\alpha_1, \alpha_2 \in \mathbb{R}$ the function $\alpha_1 z_1 + \alpha_2 z_2$ also has arithmetic mean and $\mathrm{mean}(\alpha_1 z_1 + \alpha_2 z_2) = \alpha_1 \mathrm{mean}(z_1) + \alpha_2 \mathrm{mean}(z_2)$.

Example 5.7 To be more concrete we give some examples:

(i) If z is a constant function then $\mathrm{mean}(z) = z$.

(ii) If z decays to zero, i.e., for every $\varepsilon > 0$ there exists an $R > 0$ such that $|z(x)| \leq \varepsilon$ for $x \notin B(R)$, then $\mathrm{mean}(z) = 0$.

(iii) If $z \in L^p(\mathbb{R}^d, dx)$, $p \in [1, \infty)$, then $\mathrm{mean}(z) = 0$.

(iv) Also trigonometric functions have no influence, e.g., for $d = 1$ and $z(x) = 1 + \varepsilon \sin(x)$ we have $\mathrm{mean}(z) = 1$.

(v) A less trivial example is the following one. Given $z_0, z_1 \geq 0$, let z be the function defined at each $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$ by

$$z(x) = \begin{cases} z_1 & \text{if } x_1 \geq 0 \\ z_0 & \text{otherwise} \end{cases} .$$

In this case one finds $\mathrm{mean}(z) = \frac{z_0 + z_1}{2}$. Note that in this case $\hat{z}$ is not a signed measure.

Remark 5.8 The arithmetic mean does not exist for all bounded non-negative functions.

1. For $d = 1$ and
$$z(x) = \begin{cases} 1 & \text{if } 2^{2k} \le |x| \le 2^{2k+1} \text{ for a } k \in \mathbb{N}_0 \\ 0 & \text{otherwise} \end{cases},$$
the integral in (29) oscillates between 1/3 and 2/3. Thus z does not have arithmetic mean.

2. Another example is given by the following slowly oscillating function
$$z(x) = \cos(\ln(1+|x|)) + z_0, \quad x \in \mathbb{R}^d,$$
where z_0 is a constant greater or equal to 1. Then for large R it holds
$$\frac{1}{\mathrm{vol}(B(R))} \int_{B(R)} z(x)\, dx \sim \frac{d}{\sqrt{1+d^2}} \sin\Big(\ln(R+1) + \arctan(d)\Big) + z_0.$$
In general slowly varying functions will show in general spurious behavior.

First, we prove that one may reduce the proof of Proposition 5.5 to a one particle system. This follows from the independent movement of the particles discussed in Section 3.

Lemma 5.9 *Let $0 \le z \in L^1_{\mathrm{loc}}(\mathbb{R}^d, dx)$ be such that*
$$\lim_{t\to+\infty} \int_{\mathbb{R}^d} dx\, e^{tA}\varphi(x)\ z(x) =: \int_{\mathbb{R}^d} z_\infty(dx)\varphi(x)$$
exists for all $\varphi \in \mathcal{D}(\mathbb{R}^d)$. If z_∞ is a (non-negative) Radon measure on $\mathbb{R}^d$, then the one-dimensional distribution $P^{\mathbf{X}}_{\pi_z,t}$ converges weakly to π_{z_∞} when t tends to infinity.

Proof. According to (27), for all $\varphi \in \mathcal{D}(\mathbb{R}^d)$ we have
$$\int_\Theta P^{\mathbf{X}}_{\pi_z,t}(d\gamma)\, e^{\langle\varphi,\gamma\rangle} = \exp\left(\int_{\mathbb{R}^d} dx\, e^{tA}(e^\varphi - 1)(x)\ z(x)\right),$$
with $e^\varphi - 1 \in \mathcal{D}(\mathbb{R}^d)$ too. By assumption, the latter converges when $t \to \infty$ to
$$\exp\left(\int_{\mathbb{R}^d} z_\infty(dx)\, (e^{\varphi(x)} - 1)\right).$$

By the definition (5) of a Poisson measure, observe that this is the Laplace transform of π_{z_∞}. The convergence of Laplace transforms implies weak convergence, see [Kal76, Theorem 4.2]). ■

It remains to derive the large time asymptotic for the one particle system.

Lemma 5.10 *Let $z \geq 0$ be a bounded measurable function such that its Fourier transform $\hat{z}$ is a signed measure. For all $\varphi \in \mathcal{S}(\mathbb{R}^d)$ one has*

$$\lim_{t\to+\infty} \int_{\mathbb{R}^d} dx\, e^{tA}\varphi(x)\; z(x) = (2\pi)^{-d/2}\;\hat{z}(\{0\}) \int_{\mathbb{R}^d} dx\, \varphi(x). \tag{30}$$

Proof. Through the Parseval formula and the explicit formula (9) for the semigroup $(e^{tA})_{t\geq 0}$ one finds

$$\int_{\mathbb{R}^d} dx\, e^{tA}\varphi(x) z(x) = \int_{\mathbb{R}^d} \hat{z}(dk)\, e^{t(2\pi)^{d/2}(\hat{a}(-k)-\hat{a}(0))}\hat{\varphi}(-k).$$

Since for all $k \in \mathbb{R}^d$, $\mathrm{Re}(\hat{a}(k) - \hat{a}(0)) \leq 0$ (cf. Remark 2.5), for every $t \geq 0$ we have

$$\left| e^{t(2\pi)^{d/2}(\hat{a}(-k)-\hat{a}(0))}\hat{\varphi}(-k)\right| \leq |\hat{\varphi}(-k)|, \quad \forall\, k \in \mathbb{R}^d,$$

where $\hat{\varphi} \in L^1(\mathbb{R}^d, \hat{z})$. Thus, an application of the Lebesgue dominated convergence theorem yields

$$\lim_{t\to\infty} \int_{\mathbb{R}^d} dx\, e^{tA}\varphi(x) z(x) = \int_{\mathbb{R}^d} \hat{z}(dk)\,\hat{\varphi}(-k) \lim_{t\to\infty} e^{t(2\pi)^{d/2}(\hat{a}(-k)-\hat{a}(0))},$$

where

$$\lim_{t\to\infty} e^{t(2\pi)^{d/2}(\hat{a}(-k)-\hat{a}(0))} = \left\{ \begin{array}{ll} 1, & k = 0 \\ 0, & k \neq 0 \end{array} \right. .$$

■

The next lemma shows that the existence of the arithmetic mean is stable under L^1-convergence. In particular, it yields that the condition assumed in Proposition 5.5 on the Fourier transform of the intensity is sufficient to ensure the existence of the arithmetic mean.

Lemma 5.11 *Let $z \geq 0$ be a bounded measurable function. Assume that there exist a total subset of $L^1(\mathbb{R}^d, dx)$ and a $C > 0$ such that for all φ in that total subset we have*

$$\lim_{R\to\infty} R^{-d} \int_{\mathbb{R}^d} dx\, \varphi(x/R) z(x) = C \int_{\mathbb{R}^d} \varphi(x) dx. \tag{31}$$

Then z has an arithmetic mean. In addition, equality (31) holds for $C = \mathrm{mean}(z)$ and for any $\varphi \in L^1(\mathbb{R}^d, dx)$.

Proof. Let φ be an arbitrary function in $L^1(\mathbb{R}^d, dx)$. Then for any $\varepsilon > 0$ there exists a ψ in the aforementioned total set such that $\|\varphi - \psi\|_{L^1} \le \varepsilon$ and ψ fulfills (31). By the following estimate, (31) also holds for φ:

$$\begin{aligned}
&\left| R^{-d} \int_{\mathbb{R}^d} dx\, \varphi(x/R) z(x) - C \int_{\mathbb{R}^d} dx \varphi(x) \right| \\
&\le \left| \int_{\mathbb{R}^d} dx\, (\varphi(x) - \psi(x))\, z(xR) \right| + |C| \left| \int_{\mathbb{R}^d} dx (\psi(x) - \varphi(x)) \right| \\
&\quad + \left| R^{-d} \int_{\mathbb{R}^d} dx\, \psi(x/R) z(x) - C \int_{\mathbb{R}^d} dx \psi(x) \right| \\
&\le \Big(\|z\|_u + |C| \Big) \|\varphi - \psi\|_{L^1} + \left| R^{-d} \int_{\mathbb{R}^d} dx\, \psi(x/R) z(x) - C \int_{\mathbb{R}^d} dx \psi(x) \right|.
\end{aligned}$$

Hence equality (31) holds for any $\varphi \in L^1(\mathbb{R}^d, dx)$. Applying the result obtained till now to

$$\varphi_r(x) := \begin{cases} \frac{1}{\mathrm{vol}(B(r))}, & x \in B(r) \\ 0, & \text{otherwise} \end{cases} \qquad r > 0 \tag{32}$$

yields that the l.h.s. of (31) coincides with the mean(z). ■

Corollary 5.12 *Given a bounded measurable function $z \ge 0$ the following two results hold:*

1. *If z has arithmetic mean, then the limit in (31) exists for all $\varphi \in L^1(\mathbb{R}^d, dx)$ and $C = \mathrm{mean}(z)$.*
2. *If the Fourier transform of z is a signed measure, then z has arithmetic mean and*
$$\mathrm{mean}(z) = (2\pi)^{-d/2}\ \hat{z}(\{0\}).$$

Proof. The first part is a direct consequence of Lemma 5.11 using the total set of all φ_r, defined as in (32), and their translates.

If $\hat{z}$ is a signed measure, then due to Parseval's formula for all $\varphi \in \mathcal{S}(\mathbb{R}^d)$ it holds

$$R^{-d} \int_{\mathbb{R}^d} dx\, \varphi(x/R) z(x) = \int_{\mathbb{R}^d} \hat{z}(dk) \hat{\varphi}(-kR) \to \hat{z}(\{0\}) \hat{\varphi}(0), \quad R \to \infty.$$

The limit exists because $\hat{\varphi}$ is continuous and decays quicker than any inverse polynomial. Hence, when R goes to ∞, $\hat{\varphi}(-kR)$ converges to the characteristic function of the set $\{0\}$. The family $(\hat{\varphi}(-kR))_{R\geq 0}$ is dominated by an integrable function. The second part then follows by Lebesgue's dominated convergence theorem and an application of Lemma 5.11 for the total set $\mathcal{S}(\mathbb{R}^d)$. ■

Remark 5.13 Let us underline the difference between $\hat{z}(\{0\})$ and the evaluation of $\hat{z}$ as a function at zero. We explain this for the case when z is a bounded L^1-function. The Fourier transform of z is then a continuous function which we denote for the moment by $\tilde{z}$. Evaluation at zero makes sense in this case. However, we want to consider the Fourier transform as a generalized function, i.e., as a linear form on a function space. We assume that this linear form is regular enough to be expressed by a signed measure. If, as in our case, z is an L^1-function, then its Fourier transform is the measure $\hat{z}(dk) = \tilde{z}(k)dk$. Thus $\hat{z}(\{0\}) = 0$, which does not necessarily coincides with $\tilde{z}(0) = \int_{\mathbb{R}^d} dx z(x)$.

Below we list the Fourier transforms (interpreted as generalized functions, signed measures, respectively) of the examples given in Examples 5.7, provided they exist and with the same enumeration:

(i) $(2\pi)^{d/2} z\delta_0(dk)$.

(iii) a continuous function for $p = 1$ or an $L^{p/(p-1)}$-function for $1 < p \leq 2$ multiplied in both cases by the Lebesgue measure.

(iv) $(2\pi)^{d/2}\left(\delta_0(dk) + i\varepsilon/2\delta_1(dk) - i\varepsilon/2\delta_{-1}(dk)\right)$.

(v) In the one dimensional case the Fourier transform is the following generalized function $\hat{z}(k) = \sqrt{2\pi}(z_0 + z_1)/2\delta_0(k) + i(z_0 - z_1)/\sqrt{2\pi}\mathcal{P}(1/k)$, where $\mathcal{P}(1/k)$ denotes the Cauchy principal value of $1/k$. Using this explicit formula the conclusion of Proposition 5.5 can be shown although the assumptions of Proposition 5.5 are not fulfilled.

Applying the same technique as in Lemma 5.10 we can prove that the time asymptotic depends only on the behavior of z at infinity.

Corollary 5.14 *Let $z_1, z_2 \geq 0$ be two bounded measurable functions. If $z_1 - z_2 \in L^1(\mathbb{R}^d, dx)$, then the free Kawasaki dynamics with initial distribution π_{z_1}*

and the free Kawasaki dynamics with initial distribution π_{z_2} have the same large time asymptotic limit.

Proof. For all $\varphi \in \mathcal{D}(\mathbb{R}^d)$ it follows from Lemma 5.10 and Corollary 5.12 that

$$\int_{\mathbb{R}^d} dx\, e^{tA}(e^{\varphi}-1)(x) z_1(x) - \int_{\mathbb{R}^d} dx\, e^{tA}(e^{\varphi}-1)(x) z_2(x)$$

converges when t goes to ∞ to

$$(2\pi)^{-d/2}\widehat{(z_1-z_2)}(\{0\}) \int_{\mathbb{R}^d} (e^{\varphi(x)}-1)dx.$$

As we discussed in Example 5.7, the assumption on $z_1 - z_2$ implies that $\mathrm{mean}(z_1 - z_2) = 0$. Hence

$$\lim_{t\to+\infty} \int_\Gamma P^{\mathbf{X}}_{\pi_{z_1},t}(d\gamma)\, e^{\langle\varphi,\gamma\rangle} = \lim_{t\to+\infty} \int_\Gamma P^{\mathbf{X}}_{\pi_{z_2},t}(d\gamma)\, e^{\langle\varphi,\gamma\rangle}.$$

By [Kal76, Theorem 4.2]), this is enough to show the required result. ■

Throughout this subsection, the study of the time asymptotic behavior of the free Kawasaki process with an initial distribution π_z was based on the analysis of the Laplace transform

$$\int_\Gamma \pi_z(d\gamma)\, \mathbf{E}_\gamma[e^{\langle\varphi,\mathbf{X}_t\rangle}], \tag{33}$$

cf. Lemma 5.9. Actually, we had studied the time asymptotic behavior of the so-called empirical field corresponding to a $\varphi \in \mathcal{D}(\mathbb{R}^d)$,

$$n_t(\varphi, \mathbf{X}) := \langle \varphi, \mathbf{X}_t\rangle = \sum_{x\in\mathbf{X}_t} \varphi(x).$$

The special role of the empirical fields is essential in the next subsection.

5.2 Hydrodynamic limits

In the sequel let $z \geq 0$ be a bounded measurable function. In order to obtain a macroscopic description of our system, we rescale simultaneously the empirical field $n_t(\varphi, \mathbf{X}) = \langle\varphi, \mathbf{X}_t\rangle$, $\varphi \in \mathcal{D}(\mathbb{R}^d)$, in space and in time. The

scale transformation in space is given by $\langle \varphi, \gamma \rangle \to \varepsilon^d \langle \varphi(\varepsilon \cdot), \gamma \rangle$, and in time by $t \to \varepsilon^{-\kappa} t$ for some $\kappa > 0$. To obtain non-trivial macroscopic density profiles, one has to scale the initial intensity as well, $z \to z(\varepsilon \cdot)$. This scaling yields a scaling of the Laplace transform of the empirical field, in other words, the Laplace transform of the one-dimensional distribution of the scaled process. For each $t \geq 0$ and each $\varphi \in \mathcal{D}(\mathbb{R}^d)$ we obtain from (13) the following form

$$\begin{aligned} & \int_\Gamma \pi_{z(\varepsilon\cdot)}(d\gamma)\, \mathbf{E}_\gamma \left[e^{\varepsilon^d \langle \varphi(\varepsilon\cdot), \mathbf{X}_{\varepsilon^{-\kappa} t} \rangle} \right] \\ = & \int_\Gamma \pi_{z(\varepsilon\cdot)}(d\gamma)\, e_B \left(e^{\varepsilon^{-\kappa} t A} \left(e^{\varepsilon^d \varphi(\varepsilon\cdot)} - 1 \right), \gamma \right) \\ = & \exp \left(\int_{\mathbb{R}^d} dx \, \left(e^{\varepsilon^{-\kappa} t A} \left(e^{\varepsilon^d \varphi(\varepsilon\cdot)} - 1 \right) \right) (x) z(\varepsilon x) \right). \end{aligned} \tag{34}$$

In the sequel we denote the scaled empirical field by

$$n_t^{(\varepsilon)}(\varphi, \mathbf{X}) := n_{\varepsilon^{-\kappa} t}(\varepsilon^d \varphi(\varepsilon\cdot), \mathbf{X}) = \varepsilon^d \left\langle \varphi(\varepsilon\cdot), \mathbf{X}_{\varepsilon^{-\kappa} t} \right\rangle. \tag{35}$$

According to the independent movement of the particles, we are again able to reduce the study of the infinite particle system to an effective one particle system. Again technical difficulties arise from the fact that the scale of the system size is much larger than the scale of space and time considered in the empirical field. Actually, the system size is infinite. Under the additional assumption that the Fourier transform of the activity z is a signed measure, the hydrodynamic limit can be derived rather directly using Fourier techniques, cf. Propositions 5.15 and 5.17. The general case of just bounded activities requires more technical involved considerations (postponed to Proposition 5.19).

Proposition 5.15 *Let $z \geq 0$ be a bounded measurable function such that its Fourier transform is a signed measure. For each $t \geq 0$ the following limit exists for all $\varphi \in \mathcal{D}(\mathbb{R}^d)$*

$$\lim_{\varepsilon \to 0^+} \int_\Theta \pi_{z(\varepsilon\cdot)}(d\gamma)\, \mathbf{E}_\gamma \left[e^{n_t^{(\varepsilon)}(\varphi, \mathbf{X})} \right] = \int_{\mathcal{D}'(\mathbb{R}^d)} \delta_{\rho_t}(d\omega) e^{\langle \varphi, \omega \rangle} \tag{36}$$

whenever one of the following conditions is fulfilled:

1. *If*

$$a_i^{(1)} := \int_{\mathbb{R}^d} dx\, x_i a(x) < \infty, \quad \forall\, i = 1, \cdots, d,$$

and $a^{(1)} := (a_1^{(1)}, \cdots, a_d^{(1)}) \neq 0$*, then for* $\kappa = 1$ *the limiting density* ρ_t *is given (as a generalized function) by*

$$\int_{\mathbb{R}^d} dx\, \rho_t(x)\varphi(x) := \int_{\mathbb{R}^d} dx\, z(x + ta^{(1)})\varphi(x), \quad \varphi \in \mathcal{D}(\mathbb{R}^d);$$

2. *If* $a^{(1)} = 0$*, and*

$$a_{ij}^{(2)} := \int_{\mathbb{R}^d} dx\, x_i x_j a(x) < \infty, \quad \forall\, i, j = 1, \cdots, d,$$

then for $\kappa = 2$ *the limiting density* ρ_t *is given (as a generalized function) by*

$$\int_{\mathbb{R}^d} dx\, \rho_t(x)\varphi(x) := \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} dx\, z(x) \int_{\mathbb{R}^d} dk\, e^{i\langle k,x\rangle} e^{-\frac{t}{2}\langle a^{(2)}k,k\rangle} \hat{\varphi}(k), \quad \varphi \in \mathcal{D}(\mathbb{R}^d)$$

where $a^{(2)}$ *denotes the* $d \times d$ *matrix with coefficients* $a_{ij}^{(2)}$*.*

Proof. By (34), for each $t \geq 0$ and each $\varphi \in \mathcal{D}(\mathbb{R}^d)$, one has

$$\begin{aligned}
& \int_\Gamma \pi_{z(\varepsilon\cdot)}(d\gamma)\, \mathbf{E}_\gamma \left[e^{n_t^{(\varepsilon)}(\varphi,\mathbf{X})} \right] \\
= \;& \exp\left(\int_{\mathbb{R}^d} dx\, e^{\varepsilon^{-\kappa}tA} (e^{\varepsilon^d \varphi(\varepsilon\cdot)} - 1)(x)\ z(\varepsilon x) \right) \\
= \;& \exp\left(\int_{\mathbb{R}^d} dx\, e^{\varepsilon^{-\kappa}tA} \Big(e^{\varepsilon^d \varphi(\varepsilon\cdot)} - 1 - \varepsilon^d \varphi(\varepsilon\cdot) \Big)(x)\ z(\varepsilon x) \right) \cdot \qquad (37) \\
& \cdot \exp\left(\int_{\mathbb{R}^d} dx\, \varepsilon^d z(\varepsilon x) \left(e^{\varepsilon^{-\kappa}tA} \varphi(\varepsilon\cdot) \right)(x) \right). \qquad (38)
\end{aligned}$$

Concerning (37), we observe that due to the contractivity property of the semigroup $(e^{tA})_{t\geq 0}$ in $L^1(\mathbb{R}^d, dx)$ (cf. Proposition 2.3) we have

$$\begin{aligned}
& \left| \int_{\mathbb{R}^d} dx\, e^{\varepsilon^{-\kappa}tA} \Big(e^{\varepsilon^d \varphi(\varepsilon\cdot)} - 1 - \varepsilon^d \varphi(\varepsilon\cdot) \Big)(x) z(\varepsilon x) \right| \qquad (39) \\
\leq \;& \| e^{\varepsilon^d \varphi(\varepsilon\cdot)} - 1 - \varepsilon^d \varphi(\varepsilon\cdot) \|_{L^1(\mathbb{R}^d,dx)} \|z\|_u \\
\leq \;& \varepsilon^d \|\varphi\|^2_{L^1(\mathbb{R}^d,dx)} e^{\|\varphi\|_{L^1(\mathbb{R}^d,dx)}} \|z\|_u,
\end{aligned}$$

for more details see Lemma A.1 in the Appendix A below. Thus, the proof reduces to check the existence of the limit of the exponential (38) when ε

converges to 0. For this purpose one shall apply the Parseval formula to the exponent in (38). The use of the explicit formula (9) for the semigroup $(e^{tA})_{t\geq 0}$ then yields

$$\int_{\mathbb{R}^d} \hat{z}(dk)\, e^{\varepsilon^{-\kappa}t(2\pi)^{d/2}(\hat{a}(-\varepsilon k)-\hat{a}(0))}\hat{\varphi}(-k). \tag{40}$$

Here, observe that since for all $k \in \mathbb{R}^d$, $\mathrm{Re}(\hat{a}(k)-\hat{a}(0)) \leq 0$ (cf. Remark 2.5), one has

$$\left| e^{\varepsilon^{-\kappa}t(2\pi)^{d/2}(\hat{a}(-\varepsilon k)-\hat{a}(0))}\hat{\varphi}(-k)\right| \leq |\hat{\varphi}(-k)|\,, \quad \forall\, k \in \mathbb{R}^d, \varepsilon > 0,$$

where, in particular, $\hat{\varphi} \in L^1(\mathbb{R}^d, \hat{z})$. Therefore, due to the Lebesgue dominated convergence theorem, one may infer the existence of the required limit from the existence of the limit

$$\lim_{\varepsilon\to 0^+} \frac{\hat{a}(-\varepsilon k) - \hat{a}(0)}{\varepsilon^\kappa} \tag{41}$$

for each $k \in \mathbb{R}^d$.

Case 1: Let $\kappa = 1$. Then, under the assumptions stated in 1, the function $\hat{a}$ is differentiable at 0, and thus (41) exists with

$$\lim_{\varepsilon\to 0^+} \frac{\hat{a}(-\varepsilon k) - \hat{a}(0))}{\varepsilon} = \frac{i}{(2\pi)^{d/2}}\langle k, a^{(1)}\rangle$$

for each $k \in \mathbb{R}^d$. Hence the limit (38) exists, being equal to

$$\begin{aligned}
\int_{\mathbb{R}^d} \hat{z}(dk)\, e^{it\langle k,a^{(1)}\rangle}\hat{\varphi}(-k) &= \frac{1}{(2\pi)^{d/2}}\int_{\mathbb{R}^d} dx\, z(x) \int_{\mathbb{R}^d} dk\, e^{i\langle k,(-x+ta^{(1)})\rangle}\hat{\varphi}(-k)\\
&= \int_{\mathbb{R}^d} dx\, z(x)\varphi(x-ta^{(1)})\\
&= \int_{\mathbb{R}^d} dx\, z(x+ta^{(1)})\varphi(x), \quad \varphi \in \mathcal{D}(\mathbb{R}^d).
\end{aligned}$$

Therefore, the Laplace transform of the rescaled empirical field converges to

$$\exp\left(\int_{\mathbb{R}^d} dx\, z(x+ta^{(1)})\varphi(x)\right).$$

This means that the limiting distribution is the Dirac measure with mass at $z(x+ta^{(1)})$.
Case 2: Now let $\kappa = 2$. Then (41) can be written as

$$\begin{aligned}
\frac{\hat{a}(-\varepsilon k) - \hat{a}(0)}{\varepsilon^2} &= \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} dx\, a(x) \frac{e^{i\varepsilon\langle x,k\rangle} - 1}{\varepsilon^2} \\
&= \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} dx\, a(x) \left(\frac{i}{\varepsilon}\langle x,k\rangle - \frac{1}{2}\sum_{i,j=1}^{d} x_i x_j k_i k_j + \frac{o(\varepsilon^2)}{\varepsilon^2} \right) \\
&= \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} dx\, a(x) \left(-\frac{1}{2}\sum_{i,j=1}^{d} x_i x_j k_i k_j + \frac{o(\varepsilon^2)}{\varepsilon^2} \right),
\end{aligned}$$

where we have used the Taylor expansion of the function $e^{i\langle x,k\rangle}$ at $x=0$ and the assumptions for the Case 2. Since $a \in L^1(\mathbb{R}^d, dx)$, the latter converges to $-1/2(2\pi)^{-d/2}\langle a^{(2)}k, k\rangle$, and thus

$$\begin{aligned}
&\lim_{\varepsilon\to 0^+} \int_\Gamma \pi_{z(\varepsilon\cdot)}(d\gamma)\, \mathbf{E}_\gamma \left[e^{n_t^{(\varepsilon)}(\varphi,\mathbf{X})} \right] \\
= \;& \exp\left(\int_{\mathbb{R}^d} \hat{z}(dk)\, e^{-\frac{t}{2}\langle a^{(2)}k,k\rangle} \hat{\varphi}(-k) \right) \\
= \;& \exp\left(\frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} dx z(x) \int_{\mathbb{R}^d} dk e^{-i\langle k,x\rangle} e^{-\frac{t}{2}\langle a^{(2)}k,k\rangle} \hat{\varphi}(-k) \right).
\end{aligned}$$

■

Remark 5.16 For continuous differentiable z the limiting density $\rho_t(x) = z(x+ta^{(1)})$ obtained in Proposition 5.15 is the strong solution of the linear partial differential equation $\frac{\partial}{\partial t}\rho_t(x) = \langle a^{(1)}, \nabla\rho_t(x)\rangle = \mathrm{div}(a^{(1)}\rho_t(x))$ with the initial condition $\rho_0 = z$. In the same way, the second case stated in Proposition 5.15 yields a limiting density which is the strong solution of the heat equation

$$\frac{\partial}{\partial t}\rho_t(x) = \frac{1}{2}\sum_{i,j=1}^{d} a_{ij}^{(2)} \frac{\partial^2}{\partial x_i \partial x_j}\rho_t(x)$$

with the same initial condition.

Given the function a, we may decompose a into a sum of an even function p and an odd function q, $a = p + q$. Note that one always has $p^{(1)} = 0$.

Submitting also the function a to a proper scale transformation (beyond the previous scale transformations in space and in time), one may complement the statement of Proposition 5.15. The limit (41) required in Proposition 5.15 suggests the scaling $a_\varepsilon := p + \varepsilon q$ and $\kappa = 2$. Note that the assumptions on the function a (i.e., $0 \leq a \in L^1(\mathbb{R}^d, dx)$), carry over to a_ε, $\varepsilon > 0$. Hence all considerations done in Section 2 and following sections for the one particle operator A and its underlying dynamics still hold for the operator A_ε defined by (4) with a replaced by a_ε. Denote by

$$a^{(1)} := \int_{\mathbb{R}^d} dx\, xa(x) = \varepsilon^{-1} \int_{\mathbb{R}^d} dx\, xa_\varepsilon(x) = \int_{\mathbb{R}^d} dx\, xq(x).$$

Proposition 5.17 *("weak asymmetry") Let $z \geq 0$ be a bounded measurable function such that its Fourier transform is a signed measure. Under the above conditions, if $0 \neq q^{(1)} = a^{(1)} \in \mathbb{R}^d$ and $p^{(2)}_{ij} < \infty$ for every $i, j = 1, \cdots, d$, then for each $t \geq 0$ and each $\varphi \in \mathcal{D}(\mathbb{R}^d)$ the following limit exists, and it is given by*

$$\begin{aligned} &\lim_{\varepsilon \to 0^+} \int_\Gamma \pi_{z(\varepsilon \cdot)}(d\gamma)\, \mathbf{E}_\gamma[e^{n_t^{(\varepsilon)}(\varphi, \mathbf{X})}] \\ = \quad &\exp\left(\frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} dx\, z(x) \int_{\mathbb{R}^d} dk\, e^{i\langle k,x\rangle} e^{-it\langle a^{(1)},k\rangle - \frac{t}{2}\langle a^{(2)}k,k\rangle} \hat{\varphi}(k) \right). \end{aligned}$$

Proof. Since all statements for the operator A and the corresponding process also hold for A_ε, in particular, one finds that $(e^{tA_\varepsilon})_{t\geq 0}$ is also a contraction semigroup on $L^1(\mathbb{R}^d, dx)$. Similar arguments as in the proof of Proposition 5.15 reduce the proof to the analysis of existence of the limit

$$\lim_{\varepsilon \to 0^+} \frac{\widehat{a_\varepsilon}(-\varepsilon k) - \widehat{a_\varepsilon}(0)}{\varepsilon^2} = \lim_{\varepsilon \to 0^+} \frac{\hat{p}(-\varepsilon k) - \hat{p}(0)}{\varepsilon^2} + \lim_{\varepsilon \to 0^+} \frac{\hat{q}(-\varepsilon k) - \hat{q}(0)}{\varepsilon}, \quad k \in \mathbb{R}^d.$$

■

Remark 5.18 Similarly to Remark 5.16, one may then conclude that for continuous differentiable z Proposition 5.17 leads to a limiting density ρ_t which is a strong solution of the partial differential equation

$$\frac{\partial}{\partial t}\rho_t(x) = \mathrm{div}(a^{(1)}\rho_t(x)) + \frac{1}{2}\sum_{i,j=1}^d a^{(2)}_{ij} \frac{\partial^2}{\partial x_i \partial x_j}\rho_t(x)$$

with the initial condition $\rho_0 = z$.

Proposition 5.19 *Assume that a has all moments finite. Then the results stated in Propositions 5.15 and 5.17 hold for all non-negative bounded measurable functions z.*

Proof. We note that in the first part of the proof of Propositions 5.15 and 5.17 we have only used the boundedness of z. Therefore, it remains to prove the convergence of the exponent in (38), namely,

$$\int_{\mathbb{R}^d} dx\, z(x) \left(e^{\varepsilon^{-\kappa} t A_\varepsilon} \varphi(\varepsilon\cdot)\right)\left(\frac{x}{\varepsilon}\right). \tag{42}$$

Let $\psi \in L^1(\mathbb{R}^d, dx)$ be given. Then

$$\begin{aligned} & \left|\int_{\mathbb{R}^d} dx\, z(x) \left(e^{\varepsilon^{-\kappa} t A_\varepsilon} \varphi(\varepsilon\cdot)\right)\left(\frac{x}{\varepsilon}\right) - \int_{\mathbb{R}^d} dx\, z(x) \left(e^{\varepsilon^{-\kappa} t A_\varepsilon} \psi(\varepsilon\cdot)\right)\left(\frac{x}{\varepsilon}\right)\right| \\ \leq\ & \varepsilon^d \int_{\mathbb{R}^d} dx\, |z(\varepsilon x)| \left|e^{\varepsilon^{-\kappa} t A_\varepsilon} \left(\varphi(\varepsilon\cdot) - \psi(\varepsilon\cdot)\right)(x)\right| \\ \leq\ & \|z\|_u \varepsilon^d \left\|e^{\varepsilon^{-\kappa} t A_\varepsilon} \left(\varphi(\varepsilon\cdot) - \psi(\varepsilon\cdot)\right)\right\|_{L^1(\mathbb{R}^d, dx)}. \end{aligned}$$

As $(e^{tA_\varepsilon})_{t\geq 0}$ is a $L^1(\mathbb{R}^d, dx)$-contraction semigroup, the latter can be bounded by

$$\leq\ \|z\|_u \varepsilon^d \int_{\mathbb{R}^d} dx\, |\varphi(\varepsilon x) - \psi(\varepsilon x)| = \|z\|_u \|\varphi - \psi\|_{L^1(\mathbb{R}^d, dx)}.$$

Therefore, it is enough to consider (42) for φ from a total subset of $L^1(\mathbb{R}^d, dx)$. Let us consider the set of all functions in $L^1(\mathbb{R}^d, dx)$ with Fourier transform in $\mathcal{D}(\mathbb{R}^d)$. This set is total in $L^1(\mathbb{R}^d, dx)$, because $\mathcal{D}(\mathbb{R}^d)$ is dense in $\mathcal{S}(\mathbb{R}^d)$, the Fourier transform is continuous in $\mathcal{S}(\mathbb{R}^d)$, and $\mathcal{S}(\mathbb{R}^d)$ is dense in $L^1(\mathbb{R}^d, dx)$. Let φ be such a function. In order to prove the convergence of (42) it is enough to show the convergence of $(e^{\varepsilon^{-\kappa} t A_\varepsilon} \varphi(\varepsilon\cdot))(\frac{x}{\varepsilon})$ in $L^1(\mathbb{R}^d, dx)$. For this purpose it is sufficient to show the following convergence of the Fourier transform of the latter expression,

$$e^{\varepsilon^{-\kappa} t (2\pi)^{d/2} (\hat{a}_\varepsilon(-\varepsilon k) - \hat{a}(0))} \hat{\varphi}(-k) \to e^{it\langle a^{(1)}, k\rangle - \frac{0^{2-\kappa} t}{2}\langle a^{(2)} k, k\rangle} \hat{\varphi}(-k), \quad \varepsilon \to 0 \tag{43}$$

in $\mathcal{S}(\mathbb{R}^d)$. The exponent in the l.h.s. of (43) may be written as

$$\hat{a}_\varepsilon(-\varepsilon k) - \hat{a}(0) = -\varepsilon\langle \nabla \hat{a}_\varepsilon(0), k\rangle - \frac{\varepsilon^2}{(2\pi)^{d/2}} \int_0^1 (1-s) \int_{\mathbb{R}^d} \langle k, x\rangle^2 e^{is\varepsilon\langle k, x\rangle} a_\varepsilon(x) dx ds. \tag{44}$$

The first summand on the right hand side is of order ε^κ, because $\varepsilon\langle\nabla\hat{a}_\varepsilon(0), k\rangle = \frac{-i\varepsilon^\kappa}{(2\pi)^{d/2}}\int_{\mathbb{R}^d} dx\langle k, x\rangle q(x)$. Hence all derivatives of this expression grow at most polynomially and all derivatives of third order and bigger are of order ε or smaller. Hence only derivatives of the first and second order have to be computed carefully in order to see that

$$e^{-\varepsilon^{2-\kappa}\int_0^1(1-s)\int_{\mathbb{R}^d}\langle k,x\rangle^2 e^{is\varepsilon\langle k,x\rangle}a_\varepsilon(x)dxds} - e^{-\frac{0^{2-\kappa}t}{2}\langle a^{(2)}k,k\rangle} \tag{45}$$

are polynomially bounded of order ε. Summarizing, (43) and all its derivatives converge locally uniformly. As $\hat{\varphi} \in \mathcal{D}(\mathbb{R}^d)$, (43) also converges in $\mathcal{S}(\mathbb{R}^d)$. Appendix B provides the tools for a more detailed calculation. ■

6 Non-equilibrium dynamics

In this section we widen the class of initial distributions to measures far from equilibrium, that is we consider all probability measures μ on Θ as initial distributions subject only to a mild mixing condition. This means that we consider the processes constructed as in Section 3 but not necessarily with a Poissonian initial distribution. Assuming enough mixing of the initial measure μ, namely (47), we are able to generalize, incorporating ideas from [DSS82], Proposition 5.5 in Subsection 6.1 and Proposition 5.19 in Subsection 6.2.

We formulate the mixinig requirement in terms of the second Ursel function (factorial cumulant), which can be expressed in terms of the first and second correlation function (factorial moments) defined in Subsection 3.2, namely

$$u_\mu^{(2)}(x, y) := k_\mu(\{x, y\}) - k_\mu(\{x\})k_\mu(\{y\}).$$

We denote in the following the first correlation function $x \mapsto k_\mu(\{x\})$ by ρ_μ. The condition on the second Ursel function, (47), is a rather weak mixing or decay of correlation condition. In Subsection 6.3 we show that this condition is fulfilled, in particular, by Gibbs measures in the high temperature regime. It will also hold beyond that regime, cf. e.g. [BMPK04].

6.1 Large time asymptotic

Recalling (17), the Laplace transform of the one-dimensional distribution $P^{\mathbf{X}}_{\mu,t}$ can be expressed in terms of an one particle system, i.e., for all non-negative

$f \in \mathcal{S}(\mathbb{R}^d)$

$$\int_\Gamma e^{-\langle f,\gamma\rangle} P^{\mathbf{X}}_{\mu,t}(d\gamma) = \int_\Theta e_B(e^{tA}(e^{-f}-1),\gamma)\mu(d\gamma) = \int_\Theta e^{\langle \ln(e^{tA}(e^{-f}-1)+1),\gamma\rangle}\mu(d\gamma). \tag{46}$$

Proposition 6.1 *Let μ be a measure on Θ which has first and second correlation function. Assume that μ fulfills the following mixing condition*

$$sup_{x\in\mathbb{R}^d} \int_{\mathbb{R}^d} u^{(2)}_\mu(x,y)dy < \infty. \tag{47}$$

In addition, we assume that the Fourier transform of the first correlation function ρ_μ is a signed measure. Then, the one-dimensional distribution $P^{\mathbf{X}}_{\mu,t}$ converges weakly to $\pi_{\mathrm{mean}(\rho_\mu)}$ when t tends to infinity.

Proof. Given an non-negative $f \in \mathcal{S}(\mathbb{R}^d)$ such that $-1 \le e^{-f} - 1 \le 0$, let $\varphi := 1 - e^{-f}$. Using (46) and $|e^{-x} - e^{-y}| \le |x - y|$ for $x, y \ge 0$ one obtains that

$$\begin{aligned} &\left| \int_\Theta e^{\langle \ln(e^{tA}(e^{-f}-1)+1),\gamma\rangle}\mu(d\gamma) - e^{\mathrm{mean}(\rho_\mu)\int_{\mathbb{R}^d}(e^{-f}-1)(x)dx} \right| \\ \le\ & \int_\Theta \left| \langle \ln(-e^{tA}\varphi + 1),\gamma\rangle - \mathrm{mean}(\rho_\mu)\int_{\mathbb{R}^d}\varphi(x)dx \right| \mu(d\gamma) \end{aligned}$$

As $0 \le x - \ln(x+1) \le x^2$ for $-1/2 \le x \le 0$, $\|e^{tA}\varphi\|_u$ tends to zero for $t \to \infty$, because $\hat{\varphi} \in L^1(\mathbb{R}^d, dx)$ and

$$|e^{tA}\varphi(x)| \le \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} dk e^{t(2\pi)^{d/2}\mathrm{Re}(\hat{a}(k)-\hat{a}(0))}|\hat{\varphi}(k)|.$$

and $(e^{tA})_{t\ge0}$ is an $L^1(\mathbb{R}^d, dx)$-contraction we get that

$$\begin{aligned} &\int_\Theta \left| \langle \ln(e^{tA}\varphi + 1),\gamma\rangle - \langle e^{tA}\varphi,\gamma\rangle \right| \mu(d\gamma) \le \int_{\mathbb{R}^d} \left(e^{tA}\varphi(x)\right)^2 \rho_\mu(x)dx \\ \le\ & \|e^{tA}\varphi\|_u \int_{\mathbb{R}^d} e^{tA}\varphi(x)\rho_\mu(x)dx \le \|e^{tA}\varphi\|_u \|\varphi\|_{L^1}\|\rho_\mu\|_u. \end{aligned}$$

This means it is sufficient to estimate

$$\int_\Theta \left| \langle e^{tA}\varphi,\gamma\rangle - \mathrm{mean}(\rho_\mu)\int_{\mathbb{R}^d}\varphi(x)dx \right| \mu(d\gamma)$$

$$\leq \int_\Theta \left| \langle e^{tA}\varphi, \gamma\rangle - \int_{\mathbb{R}^d} e^{tA}\varphi(x)\rho_\mu(x)dx \right| \mu(d\gamma)$$
$$+ \left| \int_{\mathbb{R}^d} e^{tA}\varphi(x)\rho_\mu(x)dx - \mathrm{mean}(\rho_\mu)\int_{\mathbb{R}^d}\varphi(x)dx \right|.$$

The last term converge to zero because of Lemma 5.10 and Corollary 5.12. Due to the decay properties of the covariance function (47)

$$\begin{aligned}
&\int_\Theta \left| \langle e^{tA}\varphi, \gamma\rangle - \int_{\mathbb{R}^d} e^{tA}\varphi(x)\rho_\mu(x)dx \right| \mu(d\gamma) \\
&\leq \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} e^{tA}\varphi(x)e^{tA}\varphi(y)u_\mu^{(2)}(x,y)dxdy + \int_{\mathbb{R}^d}\left(e^{tA}\varphi(x)\right)^2\rho_\mu(x)dx \\
&\leq \|e^{tA}\varphi\|_u\|e^{tA}\varphi\|_{L^1}\left(\sup_{x\in\mathbb{R}^d}\int_{\mathbb{R}^d} u_\mu^{(2)}(x,y)dy + \|\rho_\mu\|_u\right),
\end{aligned}$$

which implies the result, as e^{tA} is an $L^1(\mathbb{R}^d, dx)$ contraction and $\|e^{tA}\varphi\|_u$ converges to zero. ■

6.2 Hydrodynamic limits

As in Subsection 5.2 we want to study the rescaled empirical field $n_t^{(\varepsilon)}(\varphi, \mathbf{X}) = \varepsilon^d \langle \varphi(\varepsilon\cdot), \mathbf{X}_{\varepsilon^{-\kappa}t}\rangle$. Since we do not have any longer a natural parameter associated to the initial measure, one cannot formulate something like slowly varying intensities. However, one sees that a possible framework is to work with a quite arbitrary sequence of initial measures $(\mu_\varepsilon)_{\varepsilon>0}$. The main restriction on this sequence is that one has to assume a particular convergence for the first correlation measure described below in more details and the mixing condtion (47) uniformly in ε. In Corollary 6.4 we prove that these conditions are fulfilled by Gibbs measures in the high temperature regime for slowly varying intensity $z(\varepsilon\cdot)$. Furthermore, the limit is identified.

Theorem 6.2 *Assume that a has all moments finite. Let $(\mu_\varepsilon)_{\varepsilon>0}$ be a sequence of measures on Θ such that ρ_{μ_ε} is uniformly bounded in ε, the limit $\lim_{\varepsilon\to 0^+}\rho_{\mu_\varepsilon}(\{x/\varepsilon\}) =: \rho_0(x)$ exists for all $x \in \mathbb{R}^d$ and the following mixing condition*

$$\sup_{x\in\mathbb{R}^d,\varepsilon>0}\int_{\mathbb{R}^d} u_{\mu_\varepsilon}^{(2)}(x,y)dy < \infty$$

holds. Then, for each $t \geq 0$, the following limit exists for all non-negative $\varphi \in \mathcal{D}(\mathbb{R}^d)$

$$\lim_{\varepsilon \to 0^+} \int_\Gamma \mu_\varepsilon(d\gamma)\, \mathbf{E}_\gamma \left[e^{-n_t^{(\varepsilon)}(\varphi, \mathbf{X})} \right] =: \int_{\mathcal{D}'(\mathbb{R}^d)} \delta_{\rho_t}(d\omega) e^{-\langle \varphi, \omega \rangle}, \tag{48}$$

whenever one of the following conditions is fulfilled:

1. *If $a^{(1)} = (a_1^{(1)}, \cdots, a_d^{(1)}) \neq 0$, then for $\kappa = 1$ the limit (48) holds for ρ_t equal to the strong solution of $\frac{\partial}{\partial t}\rho_t(x) = \mathrm{div}(a^{(1)}\rho_t(x))$ with initial condition ρ_0.*

2. *If $a^{(1)} = 0$, then for $\kappa = 2$ the limit (48) holds for ρ_t equal to the strong solution of $\frac{\partial}{\partial t}\rho_t(x) = \frac{1}{2}\sum_{i,j=1}^d a_{ij}^{(2)} \frac{\partial^2}{\partial x_i \partial x_j}\rho_t(x)$ with initial condition ρ_0.*

3. *As in Proposition 5.17, let p and q be the even and the odd part of a. Define $a_\varepsilon = p + \varepsilon q$ and assume that $0 \neq q^{(1)} = a^{(1)} \in \mathbb{R}^d$. Then, for $\kappa = 2$, the limit (48) for the dynamics w.r.t. a_ε holds for ρ_t equal to the strong solution of $\frac{\partial}{\partial t}\rho_t(x) = \mathrm{div}(a^{(1)}\rho_t(x)) + \frac{1}{2}\sum_{i,j=1}^d a_{ij}^{(2)} \frac{\partial^2}{\partial x_i \partial x_j}\rho_t(x)$ with initial condition ρ_0.*

Proof. Given an non-negative $\varphi \in \mathcal{S}(\mathbb{R}^d)$ according to (46) one may write (48) as

$$\left| \int_\Theta \mu_\varepsilon(d\gamma)\, e^{\left\langle \ln\left(e^{\varepsilon^{-\kappa} t A_\varepsilon} (e^{-\varepsilon^d \varphi(\varepsilon \cdot)} - 1) + 1 \right), \gamma \right\rangle} - e^{-\int_{\mathbb{R}^d} \varphi(x)\rho_t(x)dx} \right| \tag{49}$$

Using $|e^{-x} - e^{-y}| \leq |x - y|$ for $x, y \geq 0$ one can bound this by

$$\int_\Theta \left| \langle \ln\left(e^{\varepsilon^{-\kappa} t A_\varepsilon} (e^{-\varepsilon^d \varphi(\varepsilon \cdot)} - 1) + 1 \right), \gamma \rangle + \int_{-\mathbb{R}^d} \varphi(x)\rho_t(x)dx \right| \mu_\varepsilon(d\gamma)$$

Let us show that it is sufficient to consider

$$\int_\Theta \left| \langle e^{\varepsilon^{-\kappa} t A_\varepsilon} \varepsilon^d \varphi(\varepsilon \cdot), \gamma \rangle - \int_{\mathbb{R}^d} \varphi(x)\rho_t(x)dx \right| \mu_\varepsilon(d\gamma), \tag{50}$$

indeed, proceeding as in the proof of Proposition 6.1 we get

$$\int_\Theta \left| \langle \ln\left(e^{\varepsilon^{-\kappa} t A_\varepsilon} (e^{-\varepsilon^d \varphi(\varepsilon \cdot)} - 1) + 1 \right), \gamma \rangle + \langle e^{\varepsilon^{-\kappa} t A_\varepsilon} \varepsilon^d \varphi(\varepsilon \cdot), \gamma \rangle \right| \mu_\varepsilon(d\gamma)$$

$$
\begin{aligned}
\leq\ & \int_{\mathbb{R}^d} \left| \ln\left(e^{\varepsilon^{-\kappa} t A_\varepsilon}(e^{-\varepsilon^d \varphi(\varepsilon\cdot)} - 1)(x) + 1 \right) - e^{\varepsilon^{-\kappa} t A_\varepsilon}(e^{-\varepsilon^d \varphi(\varepsilon\cdot)} - 1)(x) \right| \rho_{\mu_\varepsilon}(dx) \\
& + \int_{\mathbb{R}^d} \left| e^{\varepsilon^{-\kappa} t A_\varepsilon}(e^{-\varepsilon^d \varphi(\varepsilon\cdot)} - 1)(x) + e^{\varepsilon^{-\kappa} t A_\varepsilon} \varepsilon^d \varphi(\varepsilon\cdot)(x) \right| \rho_{\mu_\varepsilon}(dx) \\
\leq\ & \int_{\mathbb{R}^d} \left(e^{\varepsilon^{-\kappa} t A_\varepsilon}(e^{-\varepsilon^d \varphi(\varepsilon\cdot)} - 1)(x) \right)^2 \rho_{\mu_\varepsilon}(dx) \\
& + \int_{\mathbb{R}^d} \left| e^{\varepsilon^{-\kappa} t A_\varepsilon} \left(e^{-\varepsilon^d \varphi(\varepsilon\cdot)} - 1 + \varepsilon^d \varphi(\varepsilon\cdot) \right) \right| (x) \rho_{\mu_\varepsilon}(dx) \\
\leq\ & \left\| e^{\varepsilon^{-\kappa} t A_\varepsilon}(e^{-\varepsilon^d \varphi(\varepsilon\cdot)} - 1) \right\|_u \left\| e^{\varepsilon^{-\kappa} t A_\varepsilon}(e^{-\varepsilon^d \varphi(\varepsilon\cdot)} - 1) \right\|_{L^1} \|\rho_{\mu_\varepsilon}\|_u \\
& + \left\| e^{\varepsilon^{-\kappa} t A_\varepsilon} \left(e^{-\varepsilon^d \varphi(\varepsilon\cdot)} - 1 + \varepsilon^d \varphi(\varepsilon\cdot) \right) \right\|_{L^1} \|\rho_{\mu_\varepsilon}\|_u.
\end{aligned}
$$

The latter expression is at least of order ε^d according to Lemma A.1.

Thus it remains to consider (50) which can be bounded by

$$
\begin{aligned}
& \int_\Theta \left| \langle e^{\varepsilon^{-\kappa} t A_\varepsilon} \varepsilon^d \varphi(\varepsilon\cdot), \gamma \rangle - \int_{\mathbb{R}^d} \left(e^{\varepsilon^{-\kappa} t A_\varepsilon} \varphi(\varepsilon\cdot) \right)(x/\varepsilon) \rho_{\mu_\varepsilon}(x/\varepsilon) dx \right| \mu_\varepsilon(d\gamma) \\
& + \left| \int_{\mathbb{R}^d} \left(e^{\varepsilon^{-\kappa} t A_\varepsilon} \varphi(\varepsilon\cdot) \right)(x/\varepsilon) \rho_{\mu_\varepsilon}(x/\varepsilon) dx - \int_{\mathbb{R}^d} \varphi(x) \rho_t(x) dx \right| \qquad (51)
\end{aligned}
$$

In Proposition 5.19 we show that $e^{\varepsilon^{-\kappa} t A_\varepsilon} \varphi(\varepsilon\cdot)(x/\varepsilon)$ converges in $\mathcal{S}(\mathbb{R}^d)$ to

$$
e^{-t\langle a^{(1)}, \nabla \rangle + t 0^{2-\kappa}/2 \langle \nabla, a^{(2)} \nabla \rangle} \varphi(x).
$$

By assumption, $\rho_{\mu_\varepsilon}(\{x/\varepsilon\})$ converges, in particular, in $\|\cdot\|_{0,-d-1,2}$, cf. (62), and thus one obtains

$$
\begin{aligned}
& \lim_{\varepsilon \to 0^+} \int_{\mathbb{R}^d} e^{\varepsilon^{-\kappa} t A_\varepsilon} \varphi(\varepsilon\cdot)(x/\varepsilon) \rho_{\mu_\varepsilon}(\{x/\varepsilon\}) dx \\
=\ & \int_{\mathbb{R}^d} e^{t\left(-\langle a^{(1)}, \nabla \rangle + 0^{2-\kappa}/2 \langle \nabla, a^{(2)} \nabla \rangle\right)} \varphi(x) \rho_0(x) dx.
\end{aligned}
$$

Hence the second summand in (51) converges to zero. The first summand can be bounded by the second Ursel functions as in the proof of Proposition 6.1

$$
\begin{aligned}
& \int_\Theta \left| \langle e^{\varepsilon^{-\kappa} t A_\varepsilon} \varepsilon^d \varphi(\varepsilon\cdot), \gamma \rangle - \int_{\mathbb{R}^d} \left(e^{\varepsilon^{-\kappa} t A_\varepsilon} \varphi(\varepsilon\cdot) \right)(x/\varepsilon) \rho_{\mu_\varepsilon}(x/\varepsilon) dx \right| \mu_\varepsilon(d\gamma) \\
\leq\ & \left\| e^{\varepsilon^{-\kappa} t A_\varepsilon} \varepsilon^d \varphi(\varepsilon\cdot) \right\|_u \left\| e^{\varepsilon^{-\kappa} t A_\varepsilon} \varepsilon^d \varphi(\varepsilon\cdot) \right\|_{L^1} \left(\sup_{x \in \mathbb{R}^d, \varepsilon > 0} \int_{\mathbb{R}^d} u^{(2)}_{\mu_\varepsilon}(x, y) dy + \|\rho_{\mu_\varepsilon}\|_u \right).
\end{aligned}
$$

■

6.3 Application to Gibbs measures

In this subsection we prove that the hypothesis for the results of the previous subjection are fulfilled for a concrete class of non-equilibrium measures, namely, for Gibbs measures in the high temperature low activity regime.

In order to recall the definition of a Gibbs measure, first we have to introduce a pair potential $V : \mathbb{R}^d \to \mathbb{R} \cup \{+\infty\}$, that is, a measurable function such that $V(-x) = V(x) \in \mathbb{R}$ for all $x \in \mathbb{R}^d \setminus \{0\}$. For $\gamma \in \Gamma$ and $x \in \mathbb{R}^d \setminus \gamma$ we define a relative energy of interaction between a particle located at x and the configuration γ by

$$E(x,\gamma) := \begin{cases} \displaystyle\sum_{y\in\gamma} V(x-y), & \text{if } \displaystyle\sum_{y\in\gamma} |V(x-y)| < \infty \\ \\ +\infty, & \text{otherwise} \end{cases}.$$

A probability measure μ on Γ is called a Gibbs measure corresponding to V, an intensity function $z \geq 0$, and an inverse of temperature β whenever it fulfills the Georgii-Nguyen-Zessin equation [NZ79, Theorem 2]

$$\int_\Gamma \mu(d\gamma) \sum_{x\in\gamma} H(x,\gamma) = \int_\Gamma \mu(d\gamma) \int_{\mathbb{R}^d} dx\, z(x) H(x, \gamma \cup \{x\}) e^{-\beta E(x,\gamma)} \qquad (52)$$

for all positive measurable functions $H : \mathbb{R}^d \times \Gamma \to \mathbb{R}$. This definition is equivalent to the definition via DLR-equation, see [Geo76, NZ79, Puz81, Kun99]. We observe that for $V \equiv 0$ (52) reduces to the Mecke identity, which yields an equivalent definition of the Poisson measure π_z [Mec67, Theorem 3.1]. We also note that for either $V \equiv 0$ and z not being a constant or $V \not\equiv 0$, a Gibbs measure neither is a reversible nor an invariant initial distribution for the free Kawasaki dynamics under consideration. In order to have thermodynamical behavior we assume that V is stable, i.e., there exists a $B > 0$ such that $\sum_{\{x,y\}\subset\eta} V(x-y) \geq -B|\eta|$ for all configurations $\eta \in \Gamma_0$. Furthermore, we shall assume that the parameters β, z are small (high temperature low activity regime), i.e.,

$$\|z\|_u e^{2\beta B+1} C(\beta) < 1,$$

where $C(\beta) := \int_{\mathbb{R}^d} dx\, |e^{-\beta V(x)} - 1|$. These conditions are, in particular, sufficient to insure the existence of Gibbs measures, cf. [Rue69]. Moreover, the correlation functions corresponding to such measures exist and fulfill a Ruelle

bound defined in Section 3, and thus, as noted there, they are supported on Θ.

In the high temperature low activity regime one has rather detailed information about the Ursell functions (factorial cumulants) $u_\mu : \Gamma_0 \to \mathbb{R}$ corresponding to μ, which are bounded measurable functions (also called Ursell functions), i.e., for all $f \in \mathcal{D}(\mathbb{R}^d)$ holds

$$\int_\Gamma e^{\langle f,\gamma\rangle}\mu(d\gamma) = \exp\left(\int_{\Gamma_0}\prod_{y\in\eta}(e^{f(y)}-1)u_\mu(\eta)\lambda(d\eta)\right). \tag{53}$$

The function $x \to u_\mu(\{x\})$ coincides with the first correlation function of μ.

We present the results necessary for the following. For further details, see e.g. [DSI75], [MM91] and see also [Kun01].

The Ursell functions can be expressed in terms of a sum over all connected graphs weighted by the Meyer-functions

$$k(\xi) := \sum_{G\in\mathcal{G}_c(\xi)}\prod_{\{x,y\}\in G}(e^{-\beta V(x-y)}-1) \tag{54}$$

where $\mathcal{G}_c(\xi)$ denotes the set of all connected graphs with vertex set ξ:

$$u_\mu(\eta) := \int_{\Gamma_0}\lambda_z(d\xi)k(\eta\cup\xi)\prod_{x\in\eta}z(x). \tag{55}$$

Actually, the following bound is the key result of the cluster expansion of Penrose-Ruelle type

$$|k(\xi)| \le e^{2\beta B|\xi|}\sum_{T\in\mathcal{T}(\xi)}\prod_{\{x,y\}\in G}(e^{-\beta V(x-y)}-1), \tag{56}$$

where $\mathcal{T}(\xi)$ denotes the set of all trees with set of vertices ξ. This leads to the following integrability bound

$$\begin{aligned}&\int_{\mathbb{R}^{dn}}|u_\mu(\{x,y_1,\dots,y_n\})|z(y_1)dy_1\dots z(y_n)dy_n\\ &\le\; e^{(2\beta B+1)(n+1)}\left(\|z\|_u C(\beta)\right)^n\sum_{m=0}^{\infty}\frac{(n+m+1)!}{m!}\left(e^{2\beta B+1}\|z\|_u C(\beta)\right)^m\end{aligned} \tag{57}$$

In particular the mixing condition (47) of Proposition 6.1 and Theorem 6.2 holds.

We show that Gibbs measures in the high temperature regime with a translation invariant potential fulfill the assumptions of Proposition 6.1.

Corollary 6.3 *Let $z \geq 0$ be a bounded measurable function which Fourier transform is a bounded signed measure. Let μ be a Gibbs measure corresponding to a translation invariant potential V described above, inverse temperature β and activity z which are in the high temperature low activity regime. Then the first correlation function ρ_μ has as Fourier transform a measure and the arithmetic mean*

$$\mathrm{mean}(\rho_\mu) = \frac{1}{(2\pi)^{d/2}} \sum_{n=0}^{\infty} \frac{1}{n!} \int_{\mathbb{R}^{dn}} \overline{\hat{k_r}(p_1, \ldots, p_n)} \hat{z}(\{p_1 + \ldots + p_n\}) \hat{z}(dp_1) \cdot \ldots \cdot \hat{z}(dp_n),$$

where

$$k_r(y_1, \ldots, y_n) := \sum_{G \in \tilde{\mathcal{G}}_c} \prod_{\{x_1, x_2\} \in G} (e^{-\beta V(x_1 - x_2)} - 1).$$

Here $\tilde{\mathcal{G}}_c$ denotes the set of all connected graphs with vertex set $(0, y_1, \ldots, y_n)$. As a consequence, all assumptions of Proposition 6.1 are fulfilled.

Proof. Due to the translation invariance of V and cluster expansion one can rewrite the first correlation function as

$$\rho_\mu(x) = \sum_{n=0}^{\infty} \frac{1}{n!} \int_{\mathbb{R}^{dn}} \sum_{G \in \tilde{\mathcal{G}}_c} \prod_{\{y_1, y_2\} \in G} (e^{-\beta V(y_1 - y_2)} - 1) z(x_1 - x) \cdot \ldots \cdot z(x_n - x) z(x) dx_1 \cdot \ldots \cdot dx_n,$$

where $\tilde{\mathcal{G}}_c$ denotes the set of all connected graphs with vertex set $(0, y_1, \ldots, y_n) := (0, x_1 - x, \ldots, x_n - x)$. By (56) and (57) the function k_r is integrable, and thus $\hat{k}_r$ is a continuous function which decays to zero at infinity. Hence

$$\begin{aligned} k_r(x_1 - x, \ldots, x_n - x) &= (2\pi)^{-nd/2} \int_{\mathbb{R}^{dn}} dp_1 \ldots dp_n dp e^{ip_1 x_1} \cdot \ldots \cdot e^{ip_n x_n} e^{ipx} \\ &\quad \cdot \hat{k}_r(p_1, \ldots, p_n) \delta(p - p_1 - \ldots - p_n). \end{aligned}$$

To compute the Fourier transform of ρ_μ in the weak sense it remains to compute the Fourier transform of

$$\begin{aligned} &\varphi(x) z(x_1 - x) \cdot \ldots \cdot z(x_n - x) z(x) \\ &= (2\pi)^{-(n+2)d/2} \int_{\mathbb{R}^{dn}} \hat{z}(dp_1) e^{ip_1 x_1} \cdot \ldots \cdot \hat{z}(dp_n) e^{ip_n x_n} \, dp \, e^{ipx} \\ &\quad e^{-i(p_1 + \ldots + p_n)x} \left(\hat{\varphi} * \hat{z}\right)(p), \end{aligned}$$

for any $\varphi \in \mathcal{D}(\mathbb{R}^d)$. Summarizing, we obtain that

$$\begin{aligned}\int_{\mathbb{R}^d} dx\varphi(x)\rho_\mu(x) &= \sum_{n=0}^{\infty} \frac{1}{n!} \int_{\mathbb{R}^{dn}} \hat{z}(dp_1) \cdot \ldots \cdot \hat{z}(dp_n)\hat{z}(dp) \\ &\quad \cdot \overline{\hat{k}_r(p_1, \ldots, p_n)}\hat{\varphi}(p_1 + \ldots + p_n - p).\end{aligned}$$

■

As in Subsection 5.2, we consider initial measures with a slowly varying intensity, i.e., Gibbs measures corresponding to β, V, and $z(\varepsilon\cdot)$, which we denote by μ_ε. As $\|z\|_u$ is unchanged, all scaled measures μ_ε remain in the high temperature low activity regime and the bound (57) holds uniformly in $\varepsilon > 0$. For Gibbs measures which are not Poisson measures, the first correlation function is not any longer just the intensity. The function appearing as initial value in the limiting partial differential equation is the scaling limit of the first correlation function and not just the unscaled activity. Let us describe what is the scaling limit of the first correlation function. Denote by ρ_c^{equi} the correlation function corresponding to the Gibbs measure with constant activity c, temperature β and potential V. Due to the translation invariance of V this correlation function is a constant function. Given a function $z \geq 0$ denote by $x \mapsto \rho_{z(x)}^{\mathrm{equi}}$ the function which associates to each x the constant value of the first correlation function of the Gibbs measure with constant activity $z(x)$. This function is the scaling limit of the first correlation of μ_ε. Note that $\rho_\mu(x) = u_\mu(\{x\})$.

Corollary 6.4 *Given a bounded measurable function $z \geq 0$, a potential V, and an inverse temperature β fulfilling the conditions of the high temperature and low activity regime, the corresponding Gibbs measures fulfill all assumptions of Theorem 6.2. Moreover,*

$$\rho_0(x) := \lim_{\varepsilon \to 0^+} u_{\mu_\varepsilon}(\{x/\varepsilon\}) = \rho_{z(x)}^{\mathrm{equi}}. \tag{58}$$

Proof. According to the cluster expansion of u_μ and the translation invariance of V we obtain that

$$u_{\mu_\varepsilon}(\{x/\varepsilon\}) = \sum_{n=1}^{\infty} \frac{1}{n!} \int_{\mathbb{R}^{dn}} dy_l \sum_{G \in \mathcal{G}_c(\{0,\ldots,n\})} \prod_{\{i,j\}\in G} (e^{-\beta V(y_i - y_j)} - 1) \prod_{l=1}^{n} z(\varepsilon y_l + x), \tag{59}$$

where $y_0 := 0$. Due to (57) the above expression is uniformly integrable in ε and in x, yielding the claimed result. ■

A Estimates for hydrodynamic limits

Lemma A.1 *Let $\varphi \in \mathcal{S}(\mathbb{R}^d)$ and $0 \le \varepsilon \le 1$ be given. Then*

$$\begin{aligned} \|e^{tA}(e^{\varepsilon^d \varphi(\varepsilon \cdot)} - 1)\|_u &\le \varepsilon^d \|\varphi\|_u e^{\|\varphi\|_u} \\ \|e^{tA}(e^{\varepsilon^d \varphi(\varepsilon \cdot)} - 1)\|_{L^1(\mathbb{R}^d, dx)} &\le \|\varphi\|_{L^1(\mathbb{R}^d, dx)} e^{\|\varphi\|_{L^1(\mathbb{R}^d, dx)}} \\ \|e^{tA}(e^{\varepsilon^d \varphi(\varepsilon \cdot)} - 1 - \varepsilon^d \varphi(\varepsilon \cdot))\|_{L^1(\mathbb{R}^d, dx)} &\le \varepsilon^d \|\varphi\|^2_{L^1(\mathbb{R}^d, dx)} e^{\|\varphi\|_{L^1(\mathbb{R}^d, dx)}} \end{aligned} \tag{60}$$

Proof. On the one hand $(e^{tA})_{t \ge 0}$ is a contraction semigroup with respect to the the supremum norm. So

$$\|e^{tA}(e^{\varepsilon^d \varphi(\varepsilon \cdot)} - 1)\|_u \le \|e^{\varepsilon^d \varphi(\varepsilon \cdot)} - 1\|_u \le \sum_{n=1}^{\infty} \frac{\varepsilon^{nd}}{n!} \|\varphi\|_u^n.$$

On the other hand $(e^{tA})_{t \ge 0}$ is a contraction semigroup on $L^1(\mathbb{R}^d, dx)$. Therefore, the left-hand side of (60) is bounded by

$$\int_{\mathbb{R}^d} dx \, \frac{|e^{\varepsilon^d \varphi(x)} - 1|}{\varepsilon^d} \le \|\varphi\|_{L^1(dx)} \sum_{n=1}^{\infty} \frac{\varepsilon^{d(n-1)}}{n!} \|\varphi\|^{n-1}_{L^1(dx)} \le \|\varphi\|_{L^1(dx)} e^{\|\varphi\|_{L^1(dx)}}.$$

The last inequality follows by a similar computation. ■

B Norm estimates in $\mathcal{S}(\mathbb{R}^d)$

Let us introduce the following two equivalent systems of norms for the locally convex topological vector space $\mathcal{S}(\mathbb{R}^d)$ for $A \in \mathbb{N}$ and $M \ge 0$:

$$\|f\|_{A,M,u} := \sum_{\substack{\alpha \in \mathbb{N}_0^d \\ |\alpha| \le A}} \sup_{x \in \mathbb{R}^d} |D^\alpha f(x)| \left(1 + |x|^2\right)^M \tag{61}$$

$$\|f\|_{A,M,2} := \sum_{\substack{\alpha \in \mathbb{N}_0^d \\ |\alpha| \le A}} \left(\int_{\mathbb{R}^d} |D^\alpha f(x)|^2 \left(1 + |x|^2\right)^M dx \right)^{1/2} \tag{62}$$

Lemma B.1 *Let $f_1, f_2 \in C^\infty(\mathbb{R}^d; \mathbb{C})$ be two C^∞-functions with non-positive real part such that for an $A \in \mathbb{N}$ and a $M \ge 0$ one has $\|f_i\|_{A,-M,u} < \infty$. Then there exists a constant C depending on A and on $\|f_i\|_{A,-M,u}$ such that*

$$\|e^{f_1} - e^{f_2}\|_{A,-(A+1)M,2} \le C \|f_1 - f_2\|_{A,-M,2}. \tag{63}$$

Proof. By induction one finds

$$D^\alpha(e^f - 1)(x) = \sum_{k=1}^{|\alpha|} \sum_{\substack{\alpha_1,\ldots,\alpha_k\in\mathbb{N}^d \\ \alpha_1+\ldots+\alpha_k=\alpha}} \prod_{j=1}^{k} D^{\alpha_j} f(x)(e^{f(x)} - 1). \tag{64}$$

Thus, using telescope sums one can write the following difference as

$$\begin{aligned} D^\alpha(e^{f_1} - e^{f_2})(x) \;\; &= \;\; \sum_{k=1}^{|\alpha|} \sum_{\substack{\alpha_1,\ldots,\alpha_k\in\mathbb{N}^d \\ \alpha_1+\ldots+\alpha_k=\alpha}} \sum_{l=1}^{k} \left(\prod_{j=1}^{l-1} D^{\alpha_j} f_1(x) D^{\alpha_l} f_1(x) \prod_{j=l+1}^{k} D^{\alpha_j} f_2(x) \right. \\ &\quad \left. - \prod_{j=1}^{l-1} D^{\alpha_j} f_1(x) D^{\alpha_l} f_2(x) \prod_{j=l+1}^{k} D^{\alpha_j} f_2(x) \right) e^{f_1(x)} \\ &\quad + \prod_{j=1}^{k} D^{\alpha_j} f_2(x)(e^{f_1(x)} - e^{f_2(x)}). \end{aligned}$$

Using Cauchy-Schwarz inequality one may estimate this by

$$\begin{aligned} &\left(\int_{\mathbb{R}^d} \left| D^\alpha(e^{f_1} - e^{f_2})(x) \right|^2 \left(1 + |x|^2\right)^{-(A+1)M} dx \right)^{1/2} \\ &\leq \;\; \sum_{k=1}^{|\alpha|} \sum_{l=1}^{k} \|f_1\|_{A,-M,u}^{l-1} \|f_2\|_{A,-M,u}^{k-l-1} \|f_1 - f_2\|_{A,-M,2} \|e^{f_1}\|_u \\ &\quad + \|f_2\|_{A,-M,u}^{k} \|e^{f_1} - e^{f_2}\|_{A,-M,2} \\ &\leq \;\; |\alpha|^2 \left(1 + \|f_1\|_{A,-M,u} + \|f_2\|_{A,-M,u}\right)^{|\alpha|} \\ &\quad \left(\|f_1 - f_2\|_{A,-M,2} \|e^{f_1}\|_u + \|e^{f_1} - e^{f_2}\|_{0,-M,2} \right). \end{aligned}$$

The non-positivity of the real part of f_i and the properties of the exponential function yield

$$\|e^{f_1} - e^{f_2}\|_{0,-M,2} \leq \|f_1 - f_2\|_{0,-M,2}. \tag{65}$$

Putting the additional sum out by triangle inequality and using non-negativity one obtains

$$\|e^{f_1} - e^{f_2}\|_{A,-(A+1)M,2} \tag{66}$$

$$\leq \;\; A^3 \left(1 + \|f_1\|_{A,-M,u} + \|f_2\|_{A,-M,u}\right)^A \|f_1 - f_2\|_{A,-M,2}. \tag{67}$$

■

Lemma B.2 *Let $f, f_1, f_2 \in C^\infty(\mathbb{R}^d; \mathbb{C})$ be such that f_1, f_2 have non-positive real part and for an $A \in \mathbb{N}$ and a $M \geq 0$ one has $\|f_i\|_{A,-M,u} < \infty$ and for a $M_0 \in \mathbb{R}$ one has $\|f\|_{A,M_0,u} < \infty$. Then for any $A \in \mathbb{N}$ and for any $0 \leq M' \leq M_0 - (A+1)M$ there exists a constant C depending monotonically on A, $\|f\|_{A,M_0,u}$, and on $\|f_i\|_{A,-M,u}$ such that*

$$\|fe^{f_1} - fe^{f_2}\|_{A,M',2} \leq C\|f_1 - f_2\|_{A,-M,2}. \tag{68}$$

Proof. Due to triangle inequality and product rule one obtains

$$\left|D^\alpha\big(f(e^{f_1} - e^{f_2})\big)\right| \leq \sum_{\substack{\alpha_1,\alpha_2 \in \mathbb{N}_0^d \\ \alpha_1+\alpha_2=\alpha}} \left|D^{\alpha_1} f \; D^{\alpha_2}(e^{f_1} - e^{f_2})\right|.$$

Then using triangle inequality and the supremums norm one finds

$$\begin{aligned}
&\left(\int_{\mathbb{R}^d} \left|D^\alpha f(e^{f_1} - e^{f_2})(x)\right|^2 \left(1 + |x|^2\right)^{M_0-(A+1)M} dx\right)^{1/2} \\
&\leq \sum_{\substack{\alpha_1,\alpha_2 \in \mathbb{N}_0^d \\ \alpha_1+\alpha_2=\alpha}} \left(\int \left|D^{\alpha_1} f(x) D^{\alpha_2}(e^{f_1} - e^{f_2})(x)\right|^2 \left(1 + |x|^2\right)^{M_0-(A+1)M}\right)^{1/2} \\
&\leq A\|f\|_{A,M_0,u}\|e^{f_1} - e^{f_2}\|_{A,-(A+1)M,2}.
\end{aligned}$$

■

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